Recall: The nontrivial shape of a manifold reveals itself in several ways:

1. Violation of angle sum law, $\angle A + \beta + 90^\circ \neq 180^\circ$.
   - Can encode shape through definite angles (used in some quantum gravity approaches).

2. Violation of Pythagorean law, $a^2 + b^2 \neq c^2$.
   - Can encode shape through metric distances: $(M, g)$.

   - Can encode shape through affine connection: $(M, \Gamma)$.

Observe: Such local descriptions carry redundant information!

Why? Two (pseudo-)Riemannian manifolds $(M, g)$, $(M, g')$ must be considered equivalent, i.e., they are describing the same space-time, if there exists an isometric, i.e., metric-preserving, isomorphism:

$$\epsilon: (M, g) \rightarrow (M, g')$$

Here: $\epsilon$ is called metric-preserving if, under the pull-back map

$$T^\epsilon: T_p(M) \rightarrow T_p(M)$$

the metric changes:

$$T^\epsilon_g = g'$$

This makes it hard to identify the true degrees of freedom, so that they can be quantized.

Recall: $\epsilon$ can then be considered to be a mere change of chart.
Intuition: \((M, g), (M, g')\) that are related by an isometric diffeomorphism are mere cd change of another, i.e., have the same "shape."

Definition: A (pseudo-) Riemannian structure, say \(\mathcal{E}\), is an equivalence class of (pseudo-) Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.

\[\Rightarrow\] Space(time) will need to be modelled as a (pseudo-) Riemannian structure, \(\mathcal{E}\), i.e., as an equivalence class of pairs \((M, g)\).

Problem: These equiv. classes are hard to handle because absence or existence of \(\mathcal{E}\) is hard to check!

\[\Rightarrow\] One would like to be able to reliably identify exactly one representative \((M, g)\) per class \(\mathcal{E}\).

\[\square\] This would be called a "fixing of gauge."

\[\square\] Why would this be useful?

A key example of when gauge fixing needed: Quantum gravity

We discussed detecting and describing shape through

- defining angles
- nontrivial metric distance \((M, g)\)
- nontrivial parallel transport \((M, \Gamma)\)
Recall: Quantum theory can be formulated in path integral form.

Applied to gravity:

Expect to have to handle path integrals of the type:

\[ \int e^{iS(\Xi)} \, D\Xi \]

"all Riemannian structures \( \Xi \)"

But what we initially have is, roughly, of the form:

\[ \int e^{iS(g)} \delta(\Xi) \, Dg \quad \text{or} \quad \int e^{iS(\Xi)} \delta(\Gamma) \, D\Gamma \]

"all \( g \)" \quad "all \( \Gamma \)"

Here, \( \delta(\Xi) \) should be such that from each equivalence class of the \( g \)’s or the \( \Gamma \)’s only exactly one contributes to the path integral.

Much of quantum gravity research is concerned with working out suitable \( \delta(\Xi) \) for \( g \)’s or \( \Gamma \)’s or other variables formed from them, such as the frame fields (see "loop quantum gravity").

Q: Can one detect and describe a (pseudo-) Riemannian structure \( \Xi \) directly?

A: Possibly yes, using "Spectral Geometry":

\[ \text{Independent of coordinate systems!} \]

Idea: A manifold’s vibration spectrum \( \{ \lambda_n \} \) depends only on \( \Xi \)

Key question of the field of spectral geometry: (Weyl 1911)

Does the spectrum \( \{ \lambda_n \} \) encode all about the shape, i.e., \( \Xi \)?
Remarks:

- It cannot, if $\mathcal{M}$ has infinite volume, because then the spectrum of $\Delta$ will become (almost) completely continuous.

- The spectral geometry of pseudo-Riemannian manifolds is still very little developed.

Theorem:

- Assume $(\mathcal{M}, g)$ is a compact Riemannian manifold without boundary, $\partial \mathcal{M} = \emptyset$.

- Then, each $\text{spec}(\Delta_p)$ is discrete, with finite degeneracies and without accumulation points.

In practice:

We can describe any arbitrarily large part of the universe by a compact Riemannian manifold, $(\mathcal{M}, g)$.

This allows us to describe, e.g., 3-dim. space at any fixed time (or also 4-dim. spacetime of so-called Wightman type).

Types of waves/sounds on $\mathcal{M}$:

Consider $\rho$-form fields $w(x)$ on $\mathcal{M}$, with time evolution, e.g.:

1. Schrödinger equation: $i \hbar \partial_t w(x,t) = -\frac{\hbar^2}{2m} \Delta_p w(x,t)$

2. Heat equation: $\Delta_p w(x,t) = \partial_t w(x,t)$

3. Klein-Gordon (and acoustic) eqn: $-\frac{\hbar^2}{2m} \Delta_p w(x,t) = \beta \partial_t^2 w(x,t)$
Each of them can be solved via separation of variables:

Assume we find an eigenform \( \tilde{\omega}(x) \) of \( \Delta \) on \( M \):

\[
\Delta_p \tilde{\omega}(x) = \lambda \tilde{\omega}(x)
\]

They exist: Each \( \Delta \) is self-adjoint w.r.t. the inner product \( (\omega, \nu) = \int_M \omega \cdot \nu \).

Then:
- Schrödinger eqn solved by: \( \omega(x, t) = e^{\frac{i}{2m} \Delta t} \tilde{\omega}(x) \)
- Heat eqn solved by: \( \omega(x, t) = e^{-\frac{\Delta t}{2}} \tilde{\omega}(x) \)
- Klein Gordon eqn solved by: \( \omega_2(x, t) = e^{\pm i \sqrt{\Delta} \cdot t} \tilde{\omega}(x) \)

\( \Rightarrow \) The spectrum \( \text{spec}(\Delta_p) \) is the overtone spectrum of p-form type waves on the manifold \( M \).

Properties of \( \text{spec}(\Delta_p) \):

- Expectation:
  The spectra \( \text{spec}(\Delta_p) \) for different \( p \) carry different information about \( M \):

  E.g., scalar and vector seismic waves travel (and reflect) differently.

- But recall also:
  a) \( [\Delta, \star] = 0 \)
  b) \( [\Delta, d] = 0 \)
  c) \( [\Delta, S] = 0 \)

This will relate \( \text{spec}(\Delta_p) \) to \( \text{spec}(\Delta_{p-1}), \text{spec}(\Delta_{p+1}) \) and \( \text{spec}(\Delta_{p-1}) \):
Use $[\Delta, \ast] = 0$:

Assume: $\omega \in \Lambda^p$ and $\Delta \omega = 2\omega$.

Define: $\nu := \ast \omega \in \Lambda^{n-p}$

Then:

$$\Delta \ast \omega = \ast \Delta \omega = \ast 2\omega = 2\nu$$

$$\Rightarrow \quad \text{spec}(\Delta_p) = \text{spec}(\Delta_{n-p})$$

Next:

Careful utilization of $[\Delta, \partial] = 0$ and $[\Delta, \delta] = 0$ yields much more information about these spectra!

---

Notice that: $\Delta$ maps exact forms $\omega = d\nu$ into exact forms:

$$\Delta \omega = \Delta (d\nu) = \Delta d\nu = d\Delta \nu$$

i.e.: $\Delta : d\Lambda^p \rightarrow d\Lambda^p$

Analogously: $\Delta$ maps co-exact forms $\omega = \delta \beta$ into co-exact forms:

$$\Delta \omega = \Delta (\delta \beta) = \delta \Delta \beta$$

i.e.: $\Delta : \delta \Lambda^p \rightarrow \delta \Lambda^p$

Also: $\Delta$ can map forms into $0$, namely its eigenspace with eigenvalue 0, denoted $\Lambda^p_0$. $\Lambda^p_0$ is called the space of “harmonic” $p$-forms.

$$\Delta : \Lambda^p_0 \rightarrow 0$$
Thus: \( \Delta \) maps \( d \Lambda_\rho \) and \( \delta \Lambda_\rho \) and \( \Lambda_\rho^0 \) into themselves.

Are there any other forms that \( \Delta \) could act on? No!

**Proposition ("Hodge decomposition"):**

\[
\Lambda_\rho = d \Lambda_{\rho-1} \oplus \delta \Lambda_{\rho+1} \oplus \Lambda_\rho^0
\]

(Recall that \( \oplus \) implies that the three spaces are orthogonal!)

**Q: Why useful?**

**A:** It means that every eigenvector of \( \Delta_\rho \) is either in \( d \Lambda_{\rho-1} \), or in \( \delta \Lambda_{\rho+1} \), or in \( \Lambda_\rho^0 \) but is never a linear combination of vectors in these spaces.

**Proof:** It is clear that \( d \Lambda_{\rho-1} \subset \Lambda_\rho \) and \( \delta \Lambda_{\rho+1} \subset \Lambda_\rho \). We need to show the orthogonalities and completeness:

1. **Show that** \( d \Lambda_{\rho-1} \perp \delta \Lambda_{\rho+1} \):

   Indeed, assume \( \omega = d\varphi \in \Lambda_\rho \) and \( \beta = \delta \chi \in \Lambda_\rho \).

   Then: \( (\omega, \beta) = (d\varphi, \delta \chi) \) use \( \int_{\partial \Sigma} d\varphi \wedge \delta \chi = 0 \) √

2. **Show that** \( \omega \in \Lambda_\rho \) and \( \omega \perp d \Lambda_{\rho-1} \) and \( \omega \perp \delta \Lambda_{\rho+1} \) then: \( \omega \in \Lambda_\rho^0 \):

   Indeed, assume \( \omega \perp d \Lambda_{\rho-1} \) and \( \omega \perp \delta \Lambda_{\rho+1} \). Then:

   \( \forall \varphi: (d\varphi, \omega) = 0 \) i.e. \( -(\delta \delta \omega, \varphi) = 0 \) \( \Rightarrow \delta \omega = 0 \)

   \( \forall \chi: (\delta \chi, \omega) = 0 \) i.e. \( -(\varphi, d\omega) = 0 \) \( \Rightarrow d\omega = 0 \)

   \( \Rightarrow \Delta \omega = (d\delta + \delta d) \omega = 0 \) \( \Rightarrow \omega \in \Lambda_\rho^0 \) √
Show that if $\omega \in \Lambda^p$, then $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta \Lambda_{p+1}$.

Assume $\omega \in \Lambda^p$, i.e., $d\omega = 0$, i.e., $(d\delta + \delta d)\omega = 0$.

$\Rightarrow (\omega, (d\delta + \delta d)\omega) = 0$

$\Rightarrow (\delta\omega, \delta\omega) + (d\omega, d\omega) = 0 \Rightarrow \delta\omega = 0$ and $d\omega = 0$.

(I.e., harmonic forms are closed and co-closed but not exact or co-exact.)

Thus, $B_p = \dim(\Lambda^p_\star)$ measures topological nontriviality.

The $B_p$ are called the “Betti numbers”.

$\Rightarrow \forall \lambda \in \Lambda_{p-1} : (\lambda, \delta\omega) = 0$, i.e., $(d\lambda, \omega) = 0$.

$\Rightarrow \omega \perp d\Lambda_{p-1}$, ✓

Also: $\forall \beta \in \Lambda_{p+1} : (\beta, d\omega) = 0$, i.e., $(\delta\beta, \omega) = 0$.

$\Rightarrow \omega \perp \delta\Lambda_{p+1}$, ✓

---

**Conclusion so far:**

In the Hodge decomposition, $\Lambda_{p+1} = d\Lambda_{p-1} \oplus \delta\Lambda_p \oplus \Lambda^p_{\star}$.

$\Delta$ maps every term into

$I.\delta d\delta$, i.e., $\Delta$ can be diagonalized

in each of $\Lambda_p$, $\delta\Lambda_p$, $\Lambda^p_\star$ separately.

$\Rightarrow \Delta$ has eigenvalues and eigenfunctions on each of these subspaces, for all $\pm$:

$\text{spec}(\Delta|_{d\Lambda_p})$, $\text{spec}(\Delta|_{\delta\Lambda_p})$, $\text{spec}(\Delta|_{\Lambda^p_\star}) = \{0\}$, ...

These spectra are related!
Proposition: \[ \text{spec} \left( \Delta \right|_{d \Lambda_\nu} = \text{spec} \left( \Delta \right|_{\delta \Lambda_{\nu+1}} \right) \]

and for each eigenvector in one there is one in the other.

This means:

\[ \Lambda_{p-1} = d \Lambda_{p-2} \oplus \delta \Lambda_p \oplus \Lambda_{p-1} \]

\[ \Lambda_{p} = d \Lambda_{p-1} \oplus \delta \Lambda_{p+1} \oplus \Lambda_{p} \]

\[ \Lambda_{p+1} = d \Lambda_{p} \oplus \delta \Lambda_{p+2} \oplus \Lambda_{p+1} \]

\[ \vdots \]

Proof:

Assume: \( \lambda \in \text{spec} \left( \Delta \right|_{d \Lambda_\nu} \) with eigenvector \( w \in d \Lambda_\nu \).

Define: \( v := \delta w \in \delta \Lambda_{\nu+1} \).

Then: \( \Delta v = \Delta \delta w = \delta \Delta w = \lambda \delta w = \lambda v \)

\[ \Rightarrow \lambda \in \text{spec} \left( \Delta \right|_{\delta \Lambda_{\nu+1}} \) \text{ and } v \text{ in the eigenvector.} \]

Conversely:

Assume: \( \lambda \in \text{spec} \left( \Delta \right|_{\delta \Lambda_{\nu+1}} \) with eigenvector \( w \in \delta \Lambda_{\nu+1} \).

Define: \( v := dw \in d \Lambda_\nu \).

Then: \( \Delta v = \Delta dw = d \Delta w = \lambda dw = \lambda v \)

\[ \Rightarrow \lambda \in \text{spec} \left( \Delta \right|_{d \Lambda_\nu} \) \text{ and } v \text{ in the eigenvector.} \]
Re-use $[\delta, \ast] = 0$:

\[ \Lambda_{p+1} = d \Lambda_p \oplus \delta \Lambda_{p+1} \oplus \Lambda^\circ_p \]

\[ \Lambda_p = d \Lambda_{p-1} \oplus \delta \Lambda_p \oplus \Lambda^\circ_p \]

\[ \Lambda_{p+1} = d \Lambda_p \oplus \delta \Lambda_{p+1} \oplus \Lambda^\circ_{p+1} \]

\[ \vdots \]

Now we also found:

\[ \Lambda_p = d \Lambda_{p-1} \oplus \delta \Lambda_{p+1} \oplus \Lambda^\circ_p \]

\[ \vdots \]

\[ \Lambda_{m-p} = d \Lambda_{n-p-1} \oplus \delta \Lambda_{n-p+1} \oplus \Lambda^\circ_{n-p} \]

\[ \Rightarrow \]

Summary:

\[ \Lambda_{p-1} = d \Lambda_{p-2} \oplus \delta \Lambda_{p-1} \oplus \Lambda^\circ_{p-1} \]

\[ \Lambda_{p} = d \Lambda_{p-1} \oplus \delta \Lambda_{p} \oplus \Lambda^\circ_{p} \]

\[ \Lambda_{p+1} = d \Lambda_{p} \oplus \delta \Lambda_{p+1} \oplus \Lambda^\circ_{p+1} \]

\[ \vdots \]
Example: \( \dim(M) = 3 \)

\[
\begin{align*}
\Lambda_0 &= \delta \Lambda_1 \oplus \Lambda_0 \\
\Lambda_1 &= d \Lambda_0 \oplus \delta \Lambda_2 \oplus \Lambda_1 \\
\Lambda_2 &= d \Lambda_1 \oplus \delta \Lambda_3 \oplus \Lambda_2 \\
\Lambda_3 &= d \Lambda_2 \oplus \Lambda_3
\end{align*}
\]

Same color means same spectrum of \( \Delta \).

Conclusion: There is relatively little independent information in the spectra of \( p \)-form waves on \( M \)!

E.g., when \( \dim(M) = 3 \), then the spectrum of co-vector waves \( \text{spec}(\Delta_{|\Lambda_1}) \) has already all information of all these spectra.

---

**Literature:** (neglecting literature on detecting boundary shapes from spectra)

**Indeed:** The spectra of \( \Delta \) do not contain sufficient information in general to uniquely identify the Riemannian structure from the spectra alone:

**Examples:** Cases have been found of pairs \((M, g), (\tilde{M}, \tilde{g})\) that are isospectral for \( \Delta \) on all \( \Lambda_p \) but that are not diffeomorphically isometric!

**Nevertheless:** All examples are of limited significance:

- manifolds that are locally but not globally isometric, or
- manifolds that are isospectral only with respect to some \( \Delta \) or
- manifolds that are discrete pairs (e.g. mirror images).
Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the manifold and its spectra are given:

\[(M, g) \]

A compact Riemannian manifold \( (M, g) \) without boundary

The spectra \( \{ \lambda_m^{(i)} \} \) of Laplacians \( \Delta^{(i)} \) on the manifold.

Could be Laplacians not only on forms but also on general tensors.

Perturbation:

Now change the shape of \( (M, g) \) slightly, through:

\[ g \rightarrow g + h \]

This will slightly change the spectra to

\[ \{ \lambda_m^{(i)} \} \rightarrow \{ \lambda_m^{(i)} + \mu_m^{(i)} \} \]

Why is this linearization useful?

- One can define a self-adjoint Laplacian \( \Delta_m^{(\text{on})} \) on \( T_2(M) \), with Hilbert basis \( \{ b_m(x) \} \) and eigenvalues \( \{ \lambda_m^{(\text{on})} \} \):

\[ \Delta_m^{(\text{on})} b_m(x) = \lambda_m b_m(x) \]
The metric's perturbation $h \in T_\mu(M)$ can be expanded:

$$h = \sum_{n=1}^{\infty} h_n b_n(x)$$

The perturbation of $\text{spec}(\Delta^{(m)})$ is:

$$\{ \lambda_n \} \rightarrow \{ \lambda_n + \mu_n \} \quad \text{for} \ n \geq 1$$

We obtain a linear map $S$:

$$S: \{ h_n \} \rightarrow \{ \mu_n \}$$

$$S: h_n \rightarrow \mu_n = S_{nm} h_m$$

Notice: Consider only eigenvectors and eigenvalues up to a cutoff scale. Then, there are as many parameters $\{ h_n \}$ as $\{ \mu_n \}$.

$S$ is a square matrix.

If $\det(S') \neq 0$, then $S^{-1}$ exists.

Should be able to iterate the perturbations?

This is ongoing research.
Remarks: 

- Not all $h$ actually change the shape:

$$\text{If } h = L_\xi g, \text{ for some vector field } \xi, \text{ then } g \rightarrow g + h \text{ is merely the infinitesimal change of chart belonging to the flow induced by } \xi.$$ 

- Symmetric covariant 2-tensors such as $h$ have a canonical decomposition similar to the Hodge decomposition. Thus, $\Delta$ has three spectra on $T^2(M)$.

Reference: See also e.g. the video of my talk at P1: [http://pirsa.org/15090062](http://pirsa.org/15090062)

Infinitesimal spectral geometry arose from my paper on how spacetime could be simultaneously continuous and discrete, in the same way that information can.