Plan: I The dynamics of matter and radiation in curved spacetime
II Energy-momentum tensor
III The dynamics of spacetime itself.

Recall: On a (pseudo-)Riemannian manifold, equations are well-defined only if defined independently of any chart.

⇒ Any eqn, including the eqns of motion for matter fields must be eqns among tensors and their covariant derivatives.

⇒ Need a tensor field, \( \Psi \), for each species of particle:

\[ e^-, \ proton, \ \pi^\pm, \ photon, \ W^\pm, \ etc... \]

Notation: \( \Psi^{\alpha...\beta}_{\gamma...\delta} \) (\( i \) species label, \( j \) species label)

Note: any spinor equation can also be expressed as a (complicated) tensor equation

(see e.g. Hawking & Ellis, p 51)

Question: Could we have also an additional connection field \( \tilde{\Gamma}^{\mu}_{\nu\lambda} \)?
Yes, we can! But, the difference field $Q_{ij}^k := \Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$ is actually a tensor field!

\[
\Gamma_{\alpha \beta} \rightarrow \frac{\partial \Gamma_{\alpha \beta}^i}{\partial x^j} + \frac{\partial \Gamma_{\alpha \beta}^j}{\partial x^i} - \frac{\partial \Gamma_{\alpha \beta}^i}{\partial x^j} + \frac{\partial \Gamma_{\alpha \beta}^j}{\partial x^i} \Gamma_{ij}^k
\]

\[
\tilde{\Gamma}_{\alpha \beta} \rightarrow \frac{\partial \tilde{\Gamma}_{\alpha \beta}^i}{\partial x^j} + \frac{\partial \tilde{\Gamma}_{\alpha \beta}^j}{\partial x^i} - \frac{\partial \tilde{\Gamma}_{\alpha \beta}^i}{\partial x^j} + \frac{\partial \tilde{\Gamma}_{\alpha \beta}^j}{\partial x^i} \tilde{\Gamma}_{ij}^k
\]

\[
\Rightarrow (\Gamma_{\alpha \beta} - \tilde{\Gamma}_{\alpha \beta}) \rightarrow \frac{\partial \Gamma_{\alpha \beta}^i}{\partial x^j} + \frac{\partial \Gamma_{\alpha \beta}^j}{\partial x^i} - (\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) \Gamma_{\alpha \beta}^i
\]

\[
\Rightarrow Q_{\alpha \beta} \rightarrow \frac{\partial \Gamma_{\alpha \beta}^i}{\partial x^j} + \frac{\partial \Gamma_{\alpha \beta}^j}{\partial x^i} - \Gamma_{\alpha \beta}^k \tilde{\Gamma}_{ij}^k
\]

Introducing an additional connection $\tilde{\Gamma}$ is same as introducing simply a new tensor field $Q$.

Remark: \[\Rightarrow "\text{variations}" \delta \Gamma_{\alpha \beta} \text{ will behave tensorially!} \]

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Eqs of motion of matter fields?

**Action principle:** (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, $L$, namely a scalar function of the matter fields $\psi_{(i)}^{a \ldots b}$ and their first covariant derivatives, and now also of the metric $g$:

\[
L(\psi) = L_{\text{matter}}(\{\psi_{(i)}^{a \ldots b}\}, \{\psi_{(i)}^{a \ldots b}\}, \{\psi_{(i)}^{a \ldots b}\}, g)
\]
Define the action functional:

\[
S[\Psi] := \int_\mathcal{B} L(\Psi) V^\frac{1}{2} \, dv \in \mathbb{R}
\]

Thus, each physical field \( \Psi(x) \) (as a function of both space \( x \) and time) is mapped into a number \( S[\Psi] \).

Action principle (or postulate) of classical physics:

In nature, physical fields \( \Psi \) are such that \( S[\Psi] \) is extremal in the space of all fields \( \Psi \).

Thus: The matter fields \( \Psi \) obey:

\[
\frac{\delta S[\Psi]}{\delta \Psi} = 0 \quad (\ast)
\]

These will be the eqns of motion for the fields \( \Psi \).

Definition of (\ast)?

Def: A “variation \( S \Psi \)” of the fields \( \Psi_i(\mathbf{p}) \) in a region \( \mathcal{B} \) is a one-parameter deformation, \( \Psi_i(\lambda, \mathbf{p}) \), with \( \lambda \in (-\varepsilon, \varepsilon) \), \( \varepsilon \) deformation parameter.
so that

\[ \Psi_{ij}(0, \lambda, \rho) = \Psi_{ij}(\rho) \quad \forall \rho \in M \]

\[ \Psi_{ij}(\lambda, \lambda, \rho) = \Psi_{ij}(\rho) \quad \forall \lambda, \rho \in M - B \]

**Def:** Then, we define:

\[ \delta \Psi_{ij}(\rho) := \left. \frac{\partial \Psi_{ij}(\lambda, \rho)}{\partial \lambda} \right|_{\lambda=0} \]

**Def:** The action principle now reads:

\[ 0 = \left. \frac{\partial S[\Psi]}{\partial \lambda} \right|_{\lambda=0} \quad \text{for all variations} \ \delta \Psi_{ij} \]

**Evaluate:**

\[ 0 = \left. \frac{\partial S}{\partial \lambda} \right|_{\lambda=0} = \sum_i \int_B \left[ \frac{\partial}{\partial \Psi_{ij}^{ab \ldots \kappa}} \delta \Psi_{ij}^{ab \ldots \kappa} \right] \cdot \text{(Term I)} \]

\[ + \frac{\partial}{\partial \Psi_{ij}^{ab \ldots \kappa}} \delta \left( \Psi_{ij}^{ab \ldots \kappa} \right) \right] \sqrt{g} \ d^4 x \quad \text{(Term II)} \]

by assumption, \( L \) depends also on the 1st cov. derivatives.

**Evaluate terms I, II separately:**
Term II:

\[ \delta (\psi_{(i) \ldots (j) \ldots (d) \ldots}) = (\delta \psi_{(i) \ldots (j) \ldots d})_{i \ldots} \]

\[ \Rightarrow \text{Term II} = \sum_{i} \int_{B} \frac{\partial}{\partial \psi_{(i) \ldots (j) \ldots d}} (\delta \psi_{(i) \ldots (j) \ldots d})_{i \ldots} \sqrt{g} \, d^{4}x \]

\[ = \sum_{i} \int_{B} \left( \left( \frac{\partial}{\partial \psi_{(i) \ldots (j) \ldots d}} \right) \delta \psi_{(i) \ldots (j) \ldots d} \right)_{i \ldots} \sqrt{g} \, d^{4}x \]

One term is a "boundary term":

\[ \sum_{i} \int_{\partial B} \left( \frac{\partial}{\partial \psi_{(i) \ldots (j) \ldots d}} \right) \delta \psi_{(i) \ldots (j) \ldots d} \sqrt{g} \, d^{4}x \]

\[ = \sum_{i} \int_{\partial B} \nabla \delta \psi_{(i) \ldots (j) \ldots (d) \ldots} \]

Gauss' theorem \( \Rightarrow \)

\[ = \sum_{i} \int_{\partial B} i_{\xi} \Omega \]

Exercise:

Show that for all \( \delta \):

\[ \delta \Omega = \text{div} \delta \]

\[ \int_{\Omega} \delta \Omega = \int_{\partial \Omega} \delta \nabla \cdot \Omega \]

Recall: \( \text{div} K = \nabla \cdot \Omega \)

\[ = (i_{\xi} d + d 
abla \cdot \Omega) \]

\[ = d \nabla \cdot \Omega \]

But: \( \nabla \cdot \delta \mathbf{F} = 0 \) if \( \delta \mathbf{F} \) is constant

\[ \Rightarrow \sum_{i} \int_{\partial B} i_{\xi} \Omega = 0 \]

By Property 2) of variations.
Thus, term II simplifies and we obtain:

\[ 0 = \left. \frac{\partial S}{\partial \lambda} \right|_{\lambda=0} = \sum_i \left( \frac{\partial L}{\partial \dot{\psi}_{(i)\cdots d}} \delta \psi_{(i)\cdots d} - \left( \frac{\partial L}{\partial \psi_{(i)\cdots d}} \right)_{\text{inc}} \delta \psi_{(i)\cdots d} \right) \right|_{\Gamma} d\lambda \]

Since must hold for all variations \( \delta \psi \)

\[ \Rightarrow \]

\[ \frac{\partial L}{\partial \psi_{(i)\cdots d}} - \left( \frac{\partial L}{\partial \psi_{(i)\cdots d}} \right)_{\text{inc}} = 0 \]

"Euler-Lagrange equations"

Given \( L(\psi) \), these eqns yield the eqns. of motion for \( \psi \).

Example: A real-valued scalar field \( \psi \)

- Such \( \psi \) describe e.g.:
  - \( \pi^0 \) meson (quark + antiquark)
  - inflaton

- Lagrangian?
  - Choose geodesicods at orb. point
  - and appeal to equiv. principle.
  - Obtain from spec. relat. Lagrangian:

\[ L = -\frac{1}{2} \left( \psi_{;a} \psi_{;b} g^{ab} + \frac{m^2}{c^2} \Psi^2 \right) \]

- Euler-Lagrange equation: Klein-Gordon equation

\[ \nabla_a \psi_{;b} g^{ab} - \frac{m^2}{c^2} \Psi = 0 \]
Example: The electromagnetic fields

- Assume there are no charges (i.e., there are only EM waves)

- Define the "EM 4-potential" as a real-number-valued one-form \( A \).

- Consider the field strength tensor \( F \):
  \[
  F = dA
  \]

- Recall that the \( E \) and \( B \) fields are components of the 2-form \( F \) (up to a factor of 2).

The Lagrangian (form equiv. principle):

\[
L = -\frac{1}{16\pi} F_{ab} F^{cd} g^{ac} g^{bd}
\]

in terms of forms

Varying w.r.t. \( A \), the E.L. equations read:

\[
F_{abc} g^{bc} = 0
\]

(recall: this is \( \partial F = 0 \))

Maxwell eqns.

It is also true that

\[
F_{abc} + F_{caj} + F_{baj} = 0
\]

but this is not an Euler-Lagrange eqn. It is simply:

\[
dF = 0
\]

(Which holds because \( F = dA \) and \( d^2 = 0 \))
Example: A charged scalar field $\psi$, (such $\psi$ describe, e.g., $\pi^\pm$ mesons) together with electromagnetism.

A Equiv. principle yields from spec. relativity:

\[
L = -\frac{i}{2} (\psi^*_a - ieA_a \psi) (\psi^*_b + ieA_b \psi) g^{ab}
\]

Vary $\psi$ w.r.t. $\psi^*$ ⇒ E.L. eqn:

\[
\nabla_a g^{ab} - \frac{m^2}{\hbar^2} \psi + ieA_B g^{ab} (\psi^*_B + ieA_B \psi) + ieA_{a\beta} g^{ab} \psi = 0
\]

and varying $\psi$ w.r.t. $A_a$ yields the comple. cong. equation.

Vary $\psi$ w.r.s. to $A_a$ ⇒ E.L. eqn:

\[
\frac{1}{4\pi} F_{abc} g^{bc} - ie \psi (\psi^*_a - ieA_a \psi) + ie \psi (\psi^*_a + ieA_a \psi) = 0
\]
**Dirac equation:** (Brief treatment of basis only of Dirac spinors)

**In special relativity:** (with units such that \( c = 1 \))

\[
(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m) \psi(x) = 0
\]

(D)

where \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \) is a "Spinor"

describes spin 1/2 particles such as electrons and quarks

and the four 4x4 matrices \( \gamma^\mu \) obey:

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \delta^\mu_\nu
\]

\[\gamma^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

**Why (X)?** Equation (X) is specifically chosen so that each component of \( \psi \) obeys the Klein-Gordon equation. Indeed:

\[
(D) \Rightarrow (i \gamma^\mu \partial_\mu - m)(i \gamma^\nu \partial_\nu - m) \psi = 0
\]

\[
\Rightarrow (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2) \psi = 0
\]

\[
\text{symmetries under } \gamma^\mu \psi \Rightarrow \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2 \psi = 0
\]

\[
\Rightarrow \left( \frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right) \psi = 0
\]

(\#) \Rightarrow \left( \gamma^\mu \partial_\mu + m^2 \right) \psi = 0

which is the Klein-Gordon equation in flat space.
In general relativity:

- By choosing an orthonormal tetrad, $\xi^\mu$, we achieve
  
  \[ g_{\mu \nu} = \xi^\mu \xi^\nu \quad \forall \mu, \nu \in M \]

  i.e. one set of metrics $g_{\mu \nu}$ obeying $g_{\mu \nu} \xi^\mu \xi^\nu = 2 \delta_{\mu \nu}$ suffices.

- This motivates:
  
  \[ (i \nabla \xi^\mu \xi^\nu) \psi = 0 \]

- But what is the covariant derivative of a spinor?
  
  \[ \nabla_{\xi^\rho} \psi = ? \]

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a spinor should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector $e_\sigma$ in direction $e_\mu$:

\[ e_\sigma \rightarrow e_\sigma + \nabla_{e_\mu} e_\sigma = e_\sigma + \omega^\sigma_\mu (e_\mu) e_\sigma \]

This is an infinitesimal Lorentz transformation $\Lambda^\sigma_\mu$:

\[ e_\sigma \rightarrow \Lambda^\sigma_\nu e_\nu \quad \text{with} \quad \Lambda^\sigma_\nu = \delta^\sigma_\nu + \omega^\sigma_\nu (e_\nu) \quad \text{because } \omega^\mu_\nu \text{ obeys } \omega^\mu_\nu = - \omega^\nu_\mu \]

Which is the defining spinor for infinitesimal Lorentz transformations.
Now that we know the inf. Lorentz trans. for any inf. parallel transport:

**Strategy:** Apply the same inf. Lorentz transformation on spinors for their parallel transport.

To this end: Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

- Assume \( \{s_i\} \) are on basis in spinor space, i.e.,
  \[ \Psi = \Psi^i(x) s_i. \]

- How do the \( s_i \) transform under Lorentz transformations? i.e., what is \( \nabla_v s_i = ? \)
  (In analogy to \( \nabla_v e_\mu = \omega^\nu_{\mu}(x)e_\nu \))

From special relativity it is known that under infinitesimal Lorentz transformations,

\[ \nabla_i = s_i + \omega_i. \]

Vectors transform as

\[ e_\mu \rightarrow e_\mu + \omega_\mu e_\nu \]

and the Dirac spinors transform as:

\[ s_i \rightarrow s_i - \frac{i}{4} \omega^\nu_{\mu}[\gamma^\nu, \gamma^\mu] s_i. \]

\( \Rightarrow \) under infinitesimal Lorentz trans., the spin "rotates" by this amount.
Apply to GR:

If a vector $e_\nu$ is infinitesimally parallel transported in the direction of $e_\alpha$ then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\nu_\nu(e_\alpha)$$

which is the value of the connection 1-form, i.e.: local value of the connection form

$$e_\nu \rightarrow e_\nu + \omega^\nu_\nu(e_\alpha)e_\alpha$$

From this one can immediately read off again the covariant derivative for vectors:

$$\nabla_\alpha e_\nu = \omega^\nu_\nu(e_\alpha)e_\alpha$$

Now, when a spinor $s_\nu$ is infinitesimally parallel transported in the direction of $e_\alpha$ then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation

$$\omega^\nu_\nu(e_\alpha)$$

which is the value of the connection 1-form. Thus:

$$s_\nu \rightarrow s_\nu + \frac{1}{4} \omega(e_\alpha)_\tau\gamma [\gamma^\tau\gamma^\nu] s_\nu$$

Since, under infinitesimal parallel transport:

$$s_\nu \rightarrow s_\nu + \nabla_\nu s_\nu$$

is to be determined
The covariant derivative of the basis vectors $s_i$ of Dirac spinors is:

$$\nabla_a s_i = -\frac{i}{4} \omega_{\mu}^{\nu}(e_a) [\gamma^\mu, \gamma^\nu] s_i$$

For general Dirac spinors $\Psi(x) = \Psi'(x) s_i$, the Leibniz rule for $\nabla$ yields:

$$\nabla_a \Psi = \nabla_a (\Psi'(x) s_i) = (\nabla_a \Psi'(x)) s_i + \Psi'(x) \nabla_a s_i$$

i.e.:

$$\nabla_a \Psi = e_a(\Psi') - \frac{i}{4} \omega(e_a)^{\nu\rho} [\gamma^\rho, \gamma^\nu] \Psi$$

$e_a(\Psi)' = s_i e_a(\Psi')$ vector field

**Dirac equation:**

The general relativistic Dirac equation:

$$(i \gamma^\mu \nabla_\mu - m) \Psi = 0$$

now takes this explicit form:

$$i \gamma^\mu e_{\mu}'(\Psi') - i \frac{1}{4} \omega(e_{\mu})^{\nu\rho} \gamma^\nu [\gamma^\rho, \gamma^\mu] \Psi - m \Psi = 0$$

**Remark:** The relationship between the Dirac operator $D = i \gamma^\mu \nabla_\mu$ and the Laplace or d'Alembert operator $\Delta$ also becomes:

$$D = d + \delta$$

To this end, one re-interprets the Grassmann algebra of differential forms as a so-called Clifford algebra.