On $T^\mu{}\nu$, continued:

Recall: □ We defined $T^\mu{}\nu$ as that tensor which for all $\delta g_{\mu\nu}(x)$:

$$\frac{dS}{d\lambda|_{\lambda=0}} = \frac{1}{2} \int \left( T^\mu{}\nu \delta g_{\mu\nu} T^\nu{}_{\nu} \right) d^nx$$

□ The above is meant when writing:

$$T^\mu{}\nu = \frac{2}{V} \frac{\delta S}{\delta g_{\mu\nu}}$$

□ We found that $T^\nu{}_{\nu} = 0$ always holds.

(Since is consequence of diffeomorphism invariance)

However: □ $T^\nu{}_{\nu} = 0$ is not a conservation law!

Why? $T^\nu{}_{\nu} V^\nu$ is not a divergence, unlike $K^\nu{}_{\nu} V^\nu = d\omega_k$.

Except: if space-time possesses isometries, i.e., covariant so-called Killing fields $\xi^\mu$, i.e., fields obeying:

$$L_\xi g = 0, \ i.e., \ \xi_{\mu,\nu} = -\xi_{\nu,\mu}$$

Because then: $P^\mu{}_{\nu} = T^\mu{}_{\nu} \xi^\nu$ obeys $P^\nu{}_{\nu} = 0$

Thus:

$$\int \left( P^\mu{}_{\nu} V^\nu \right) d^nx = \left( \text{bound} \right) \int d^nx \omega_\nu = 0$$

Proposition: maximal number of indep. Killing vector fields on spacetime: 10

Actual spacetime has no Killing vector fields, but realistic simplified models of parts or all of spacetime often do:
Definition: A space-time \((M,g)\) is called "stationary" if it possesses energy conservation, i.e., if it possesses a time-like Killing vector field, i.e., if it possesses a field \(\xi\) which obeys:

\[
\xi g = 0 \quad \text{and} \quad \xi^\rho \xi_\rho = g(\xi,\xi) < 0
\]

\(\xi\) would be called a "null" or light-like Killing vector field.

Since \(\xi\) is timelike, observers can travel along the integral curves of \(\xi\) and set up a coordinate system with their own time as the time coordinate.

In such a "comoving coordinate system":

\[
\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

\(\Rightarrow\) \(0 = \xi g = \xi^\rho g_{\rho\nu} \xi_\rho + g_{\nu\rho} \xi^\rho \xi_\nu \) becomes:

\[
0 = g_{\nu,\rho} = \partial_\nu g_{\rho\nu}, \text{ i.e., } g_{\nu\rho}(x) = \text{constant in time}.
\]

\(\Rightarrow\) In static space-times, one can find a (so-called comoving) coordinate system, in which:

\[
\frac{\partial^2}{\partial x^2} g_{\nu\rho}(x^0, x^1, x^2, x^3) = 0
\]

\(\Rightarrow\) Comoving observers, i.e., those observers who travel the flow generated by \(\xi\), see no change in the prevailing local curvature.

\(\square\) However: Stationarity does not imply that there is a coordinate system in which

\[
g = \begin{pmatrix} g_{\rho\nu} & 0 & 0 & 0 \\ 0 & g_{\alpha\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

Example: The \(g\) of a stationary black hole that is rotating, given by the "Kerr metric".
**Definition:** A space-time is called "static", if the time-like Killing field $\xi$, viewed as a 1-form,

$$\xi = \xi^\mu dx^\mu$$

also obeys the "Frobenius condition":

$$\xi \wedge d\xi = 0 \quad (\forall)$$

Excuse: write it out in coordinates.

**Significance?**

$(\forall)$ holds $\Leftrightarrow g = \left(\begin{smallmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi \end{smallmatrix}\right)$ in suitable coords.

1. If, in a suitable coordinate system, $g$ is of the form $(\ast)$ and the time-like $\xi$

in $\xi = g_{\mu\nu} dx^\mu$, then $\xi \wedge d\xi = 0$ trivially.

2. One can also show that, conversely, $(\exists)$ implies existence of coords in which $(\ast)$ holds.

---

**Generic properties of $T^{\mu\nu}$:**

- $T^{\mu\nu}$ has contributions from known and also from as yet unknown matter fields (e.g., from dark matter).

- Thus, in order to draw genuine conclusions about, e.g.,

  a) the occurrence of singularities, or (Note: black hole formation stops energy dropping)

  b) the overall positivity of the energy (despite universal attraction!)

one needs plausible conjectures about the full $T^{\mu\nu}$.

**The "weak energy condition":**

$$T^{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: g(v,v) < 0$$

**Why assume it?**

(by continuity it then also holds for lightlike $v$)

Note: Negative energy would be anti-gravitating (i.e., repulsive)

All observers travel with a time-like tangent $v$.

They then see a positive local energy density $T_{\mu\nu} v^\mu v^\nu > 0$
The "dominant energy condition":

\[ T_{\mu \nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v \] (i.e., weak energy condition)

and

\[ K_\mu := T_{\mu \nu} v^\nu \text{ always } K_\mu K^\mu \leq 0 \quad \text{(i.e., } T_{\mu \nu} \text{ is non-space-like)} \]

Why assume it? a The local energy-momentum flow vector, \( K \), may not be conserved but should be non-space-like: "All flow should be into the future."

a In an orthonormal basis, the dominant energy condition takes the form:

\[ T^{00} > \left| T^{ab} \right| \]

i.e. "energy dominates over momentum."

---

The dynamics of space-time:

a Consider the full matter action:

\[ S' [g, \psi] = \int_M L(g, \psi) \sqrt{g} \, d^4 x \]

all matter

fields: e.g., photons, quarks, gluons, etc.

a The equations of motion of matter fields are

\[ \frac{\delta S'}{\delta \psi^{(i)}_{\alpha \beta}} = 0 \]

i.e.:

\[ \frac{\partial L}{\partial \psi^{(i)}_{\alpha \beta}} = \left( \frac{\partial L}{\partial \psi^{(i)}_{\alpha \beta}} \right)_{\psi^{(i)}_{\alpha \beta} = 0} \]
Do we obtain suitable equations of motion for $g_{\mu \nu}$ by setting

\[ \frac{\delta S}{\delta g_{\mu \nu}} = 0 \quad ? \]

Apparently not, because it would mean:

\[ \frac{\delta S}{\delta g_{\mu \nu}} = \frac{1}{2} T_{\mu \nu} \sqrt{g} = 0 \quad ! \]

Thus, the universe would have to be empty of matter (assuming all matter has positive energy).

Andrei Sakharov (1968):

The quantum effects of matter induce suitable extra terms in the action!

---

Sakharov's reasoning: (modernized version)

- Classical deterministic evolution of matter obeys:

  \[ \frac{\delta S}{\delta \Psi_{(i)}} = 0 \]

- But quantum theory allows every evolution $\Psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

  \[ \text{probamp.} [\Psi_{(i)}] = N e^{-\frac{S}{\hbar}} \]

- As usual in quantum theory, the actual or "effective" matter evolution $\langle \Psi_{(i)}(x,t) \rangle$ is close
to but not identical to the classical matter evolution \( \psi_i(x,t) \).

**Why?** Path integral picture: The field evolutions will close-to-extremal actions have very similar values \( \mathcal{E}_i \), because for tiny \( \delta \psi \), i.e., their path amplitudes add up. Other matter evolutions \( \psi_j(x,t) \) have widely different \( \mathcal{E}_j \), so their probabilities away another away.

Heisenberg picture: The (field) operators obey formally the same equations of motion as do the classical fields. Because of the commutation (and uncertainty) relations, however, the (field) operator expectation values generally do not obey the classical equations of motion.

- **Thus, the effective quantum fields obey equations of motion that are somewhat modified!**

  \[ \Rightarrow \text{Aim: Calculate the “effective action”} \]

  \[ S_{eff}[g,\psi] \]

  which yields the effective evolution of matter fields when matter quantum effects are taken into account.

- **Problems:**
  - These calculations are very difficult.
    \[ \Rightarrow \text{Use perturbative methods.} \]
There occur integrals that are divergent at short distances.

\[ \Rightarrow \text{Introduce a cutoff at some minimum length } \ell_c \text{ (or maximum momentum } \frac{p}{c} \text{).} \]

But as always in quantum theory, the effective action will contain terms of all possible forms that are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):

\[ S_{\text{eff}}[g,\chi] = \int \left( L + L_{\text{quintessence}} + c_1 \chi + c_2 R + c_3 \Theta(R^2) \right) \sqrt{g} \, d^4x \]

The only question is which prefactors these terms will have.

Quantum "vacuum energy" of matter

This is the local change of the vacuum energy due to quantum deformations of the quantum harmonic oscillators of the field modes.

The constants \( c_1, c_2, \text{ etc.} \) depend on:

- the details of the matter action \( L_{\text{matter}} \).
- (Bosons and Fermions tend to contribute with opposite signs)
- the order of perturbation
- the value of the short-distance cutoff:

\[ c_1 = \lambda_1 \ell_c^{-4} \quad (\text{must make up for } \text{d}^4x/x^4 \text{ from } d^4x) \]
\[ c_2 = \lambda_1 \ell_c^{-2} \quad (\text{because } R \text{ has units } [\text{length}]^{-2}) \]
\[ c_3 = \lambda_3 \ell_c^2 \quad (\text{terms } R^2 \text{ or } R^n \text{ of } R_{\mu \nu} \text{ have units } [\text{length}]^{n+1}) \]
\[ c_4 = \lambda_4 \ell_c \quad (\text{higher powers in } R \text{ are prefactors } \ell_c \text{ independent }) \]

\[ \Rightarrow \text{ For small } \ell_c, \text{ we have:} \]

\[ c_1 \gg c_2 \gg c_3 \gg c_4 \gg \ldots \]
Consider the lowest order terms:

\[ S_{\text{eff}}[g, V] = \int M (L + c_1 + c_2 R) \sqrt{g} \, d^4 x \]

and postulate now that the equations of motion for the metric follow from the action principle:

\[ \frac{\delta S_{\text{eff}}[g, V]}{\delta g_{\mu \nu}} = 0 \]

Einstein had postulated the same action principle!

We note that:

Every (effective) quantum field theory with minimum length induces Einstein gravity.

See, e.g., review: gr-qc/0204062

The equations of motion for \( g \):

The action principle, \( \frac{\delta S_{\text{eff}}}{\delta g_{\mu \nu}} = 0 \), yields:

\[ \frac{\delta}{\delta g_{\mu \nu}} \int B \left( c_1 + c_2 R_{\mu \nu} \sqrt{g} \right) \sqrt{g} \, d^4 x = - \frac{1}{2} \sqrt{g} \, T^{\mu \nu} \]

in principle, it is the effective quantum expectation value

Evaluate the left hand side:

\[ \frac{\delta}{\delta g_{\mu \nu}} \int B c_1 \sqrt{g} \, d^4 x = \int B \frac{\delta}{\delta g_{\mu \nu}} \frac{\delta}{\delta g_{\mu \nu}} \sqrt{g} \, d^4 x \]

\[ = \int B c_1 \frac{1}{2} \sqrt{g} \, \sqrt{g} \, g_{\mu \nu} \, d^4 x \]
b) \[ S \int_B c_2 \, R_{\mu \nu \alpha \beta} g^{\alpha \beta} V^8 \, d^4 x \]

\[ = \int_B c_2 (\delta R_{\mu \nu}) g^{\mu \nu} V^8 \, d^4 x + \int_B c_2 \, R_{\mu \nu} \delta (g^{\alpha \beta} V^8) d^4 x \]

\[ \text{Term I} \quad \text{Term II} \]

**Proposition**: Term I = 0

**Proof**: Choose origin of geodesic coordinates system

\[ R_{\mu \nu} = \Gamma^d_{\mu \nu, \alpha} - \Gamma^d_{\mu \nu, \alpha} + \Gamma^d_{\mu \nu, \alpha} + \Gamma^d_{\mu \nu, \alpha} \]

\[ \text{vanish at origin because } \Gamma = 0 \]

Thus:

\[ \delta R_{\mu \nu} = (\delta \Gamma^d_{\mu \nu})_{,\alpha} - (\delta \Gamma^d_{\mu \nu})_{,\alpha} \]

\[ \Rightarrow g^{\mu \nu} \delta R_{\mu \nu} = g^{\mu \nu} (\delta \Gamma^d_{\mu \nu})_{,\alpha} - g^{\mu \nu} (\delta \Gamma^d_{\mu \nu})_{,\alpha} \]

\[ = w^d_{,\alpha} \text{ for } w^d = g^{\mu \nu} \delta \Gamma^d_{\mu \nu} - g^{\mu \nu} \delta \Gamma^d_{\mu \nu} \]

Recall: \( g^{\mu \nu} = 0 \) here.

Thus, in arbitrary coordinate system:

\[ g^{\mu \nu} \delta R_{\mu \nu} = w^d_{,\alpha} \]

\[ \Rightarrow \int_B c_2 \, g^{\mu \nu} \delta R_{\mu \nu} V^8 \, d^4 x = \int_B w^d_{,\alpha} V^8 \, d^4 x \]

\[ = \int_B \text{div}_v \Omega \]

\[ = 0 \text{ on } \partial B \text{, assuming } \delta g \text{ and } \delta g_{\mu \nu} = 0 \text{ on } \partial B. \]

\[ \Rightarrow \int_{\partial B} i_w \Omega = 0 \text{ on } \partial B. \]

\[ = 0 \]
c) Evaluate term II:

$$\int_B c_2 R_{\mu \nu \sigma \delta} \delta g^{\mu \nu} d^4 x = ?$$

We have:

$$\delta (g^{\mu \nu} V_3) = (\delta g^{\mu \nu}) V_3 + g^{\mu \nu} \frac{\partial}{\partial g^{\mu \nu}} \delta g_{\alpha \beta}$$

$$= -g^{\mu \nu} \delta g_{\alpha \beta} V_3 + g^{\mu \nu} \frac{\partial}{\partial g^{\mu \nu}} \delta g_{\alpha \beta}$$

$$\Rightarrow$$

$$\int_B c_2 R_{\mu \nu \sigma \delta} \delta g^{\mu \nu} d^4 x = -\int_B c_2 \left( + R^{\alpha \beta} - \frac{1}{2} g^{\alpha \beta} R \right) V_3 \delta g_{\alpha \beta} d^4 x$$

$$\text{mark} = G^{\alpha \beta}$$

"Constant tensor"

Bringing together a) + b) + c) $$\Rightarrow$$

$$\delta \int_B \left( c_1 + c_2 R_{\mu \nu} g^{\mu \nu} \right) V_3 d^4 x$$

$$= \int_B \left( c_1 + c_2 \frac{1}{2} g^{\mu \nu} - c_2 \frac{\partial}{\partial g^{\mu \nu}} \right) V_3 \delta g_{\alpha \beta} d^4 x$$

as in the case of the T occurring calculation, one could add an antisymmetric part here and it would drop from the integrand.

$$\Rightarrow$$

$$\frac{\delta}{\delta g^{\mu \nu}} \int_B \left( c_1 + c_2 R_{\mu \nu} g^{\mu \nu} \right) V_3 d^4 x$$

$$= \left( c_1 + c_2 \frac{1}{2} g^{\mu \nu} - c_2 \frac{\partial}{\partial g^{\mu \nu}} \right) V_3$$

Remark: Choosing to add an antisymmetric part would make $G^{\mu \nu}$ and $R^{\mu \nu}$ non-symmetric, in violation of $g_{\mu \nu} g = 0$. 
Finally, we conclude:

\[ \frac{\delta S'}{\delta g^{\mu\nu}} = 0 \]

leads to this equation of motion for g:

\[ \left( \frac{1}{2} c_1 g^{\mu\nu} - c_2 g^{\mu\nu} \right) \nabla^2 g^{\mu\nu} = -\frac{1}{2} \nabla^2 T^{\mu\nu} \]

i.e.:

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{c_1}{2c_2} g^{\mu\nu} = \frac{1}{2c_2} T^{\mu\nu} \]

As is well-known, comparison with experiment requires:

\[ c_2 = \frac{1}{16\pi G} \]

Newton's constant.

⇒ **Einstein equation:**

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - 8\pi G c_1 g^{\mu\nu} = 8\pi G T^{\mu\nu} \]

\( \Lambda \) "cosmological constant"

**What is \( \Lambda \)?**

Recall: The \( \Lambda \) are constants depending on other scales which depend on the matter Lagrangian.

**First find \( \Lambda \):**

Given that \[ \frac{1}{16\pi G} = c_2 = \lambda_\Lambda \lambda_\Lambda^{-2} \]

we obtain:

\[ \Lambda_\Lambda = \sqrt[3]{16\pi G} = \sqrt[3]{16\pi \cdot 6.67 \times 10^{-11}} \]

Recall: The \( \lambda \) are some numbers of order \( 10^{-35} \), depending on details which sort of particles are in the matter content and their quark effects.

The "Planck length"
We note, therefore: When quantum field theories are cut off at about the Planck length they induce gravity with the correct cons of motion and coupling strength!

- **Now find $c_1$:**

  Given that $l_p \approx 10^{-33} \text{ m}$, quantum field theories generate a value of $c_1$, i.e., a cosmological constant of about:

  $$c_1 = \frac{2}{3} \frac{l_p^{-4}}{\hbar}$$

- **Finally, find $\Lambda$:**

  $$\Lambda_{\text{theory}} = -8\pi G c_1 = -8\pi G \lambda l_p^{-4}$$

  $$\approx 6 \times 10^{-2} \approx 10^{-2}$$ (using $\lambda \approx 1$ and $G \approx l_p^{-2}$)

  $$\approx 10^{70} \text{ m}^{-2}$$

**This is the worst physical prediction ever:**

- **Experiment:**

  Based on cosmic microwave background data and on supernova brightness versus redshift data:

  $$\Lambda_{\text{experiment}} \approx 10^{-52} \text{ m}^{-2}$$

  i.e., $\frac{\Lambda_{\text{theory}}}{\Lambda_{\text{experiment}}} \approx 10^{122}$

  **⇒ For some unknown reason, the constant part, $c_1$, of the vacuum energy of quantum field theories does essentially not contribute - where the disturbance through curvature, $c_1 R$, is real: it induces regular gravity.**