Recall: If we choose the bases $\frac{\partial}{\partial x^i}$, $\{dx^i\}$, then:

$$E.g.: \quad L_{\text{EM}} = -\frac{1}{16\pi G} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{S}'[g_{\mu\nu}, \Psi] = \int \left( \frac{1}{16\pi G} R(g_{\mu\nu}(x)) + L_{\text{matter}} (g_{\mu\nu}(x), \Psi(x), \Psi'(x)) \right) g^{\mu\nu} dx$$

$$\frac{\delta S'}{\delta \psi^{\mu\nu}} = 0 \quad \Rightarrow \quad \text{Eqs. of motion of matter}$$

$$(\text{Maxwell, Klein Gordon, eqns. etc})$$

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad \text{Einstein equations:}$$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$$

What is the Einstein equation when using a frame so that

$$g_{\mu\nu}(x) = \gamma_{\mu\nu}$$?

Recall:

Frames $\{\Theta^i\}, \{e_i\}$:

Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$, the canonical bases $\{dx^i\}$ and $\{\frac{\partial}{\partial x^i}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \ldots, x^3)$.

Thus, when changing coordinate system, $x \rightarrow x'$, one also usually automatically changes basis in $T_p(M), T_p(M)'$. 


**Important:** The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-)tangent spaces, namely from one canonical basis to another canonical basis, when we change coor. system.

Recall that a fixed vector has different coefficients in different basis:

\[ \left( \begin{array}{c} \xi^2_{x^2} = \xi^2_{y^2} = \xi^2_{z^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\xi^2_{x^2} = \xi^2_{y^2} = 0 \\
\end{array} \right) \quad \Rightarrow \quad \xi^2 = \xi^2_{x^2} = \xi^2_{y^2} = \xi^2_{z^2} = 0 \]

We notice: If we choose a fixed basis, say \( \{\Theta^i, \{e_i\} \) then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: \[ \xi^2 = \xi^2_{x^2} e_i \] coordinate system.

**Conversely:** Even staying with one coordinate system, we can freely change our choice of basis in the (co-)tangent spaces:

\[ \Theta^\nu = A^\nu_{\mu} \Theta^\mu \]
\[ e^\nu_{\mu} = (A^{-1})_{\mu}^\nu e_\nu \]

So we have e.g.:

\[ \xi^2 = \xi^2_{x^2} e_\nu = \xi^2_{x^2} A^\nu_{\mu} e_\nu = \xi^2_{x^2} e_\nu \]

i.e.:

\[ \xi^2_{x^2} = A^\nu_{\mu} \xi^2_{x^2} \]

**Examples:**

- The curvature form: \[ \Omega^\nu_{\mu} = A^\nu_{\alpha} (A^{-1})_{\mu}^\beta \Omega^\alpha_{\beta} \]
- But: the connection form \( \omega^\nu_{\mu}(s) = g^\nu_{\mu} \Gamma^\alpha_{\mu} \) obeys:

\[ \omega^\nu_{\mu} = A^\nu_{\alpha} \omega^\alpha_{\beta} (A^{-1})_{\mu}^\beta - (dA)^\nu_{\alpha} (A^{-1})_{\mu}^\beta \]

\[ = A^\nu_{\alpha} \omega^\alpha_{\beta} (A^{-1})_{\mu}^\beta - dA^\nu_{\alpha} (A^{-1})_{\mu}^\beta \]
How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

\[ \theta^i(x) = A^i_j(x) \, dx^j \]

Note: the \( A^i_j(x) \) change nontrivially when changing the coordinate system!

Our choice now: orthonormal frames, or "Tetrad's."

- We say that a frame \( \{ \theta^1, \theta^2, \theta^3, \theta^4 \} \) is orthonormal if in this frame, for all \( p \in M \):

\[ g(e_\mu, e_\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

i.e. \( g = -\theta^0 \theta^0 + \theta^i \theta^i \) for each \( i \).

Existence? Always: At each \( p \in M \) may choose \( \theta^\nu = dx^\nu \) where \( dx^\nu \) are canonical ON basis at centre of yoked fields.

Existence? Always: At each \( p \in M \) may choose \( \theta^\nu = dx^\nu \) where \( dx^\nu \) are canonical ON basis at centre of yoked fields.

Uniqueness?

For a given space-time, \((M, g)\), any ON frame yields a new ON frame by transforming the bases through

\[ \theta'^{\mu}(x) = \Lambda(x)^\mu_\nu \, \theta^\nu(x), \]

if the linear maps \( \Lambda(x) \) preserve the orthogonality:

\[ \eta_{\mu \nu} \theta^\mu \theta^\nu = \eta_{\mu \nu} \theta^\mu \theta^\nu \]

i.e. \( \eta_{\mu \nu} \Lambda^a \Lambda^b \eta_{\mu \nu} = \eta_{\mu \nu} \Lambda^a \Lambda^b \eta_{\mu \nu} \)

⇒ Frames are unique up to local Lorentz transformations.
Re-express the degrees of freedom:

- We used to specify space-times through these data: \((M, g)\).
- Now, let us specify space-times, equivalently, through data \((M, \{\Theta^i\})\):

**Namely:**

Assume the \(\{\Theta^i\}\) are given w. r. t. a basis \(\{dx^i\}\), through functions \(A^i_\nu\):

\[
\Theta^i(x) = A^i_\nu(x) dx^\nu
\]

so that: \(g_{\mu\nu}(\cdot^{0}, \cdot^{0}) = g_{\mu\nu}\) in the basis \(\{\Theta^i\}\)!

**Notice:** knowing the \(A^i_\nu(x)\), we can reconstruct \(g_{\mu\nu}(x)\) in basis \(\{dx^i\}:

We use that the abstract \(g\) is the same in every basis:

\[
g = \eta_{\mu\nu} \Theta^\nu \otimes \Theta^\mu = \eta_{\mu\nu} A^\mu_\alpha A^\nu_\beta dx^\alpha \otimes dx^\beta = g_{\mu\nu}(x) dx^\alpha \otimes dx^\beta
\]

\[
\Rightarrow g_{\alpha\beta}(x) = \eta_{\mu\nu} A^\mu_\alpha(x) A^\nu_\beta(x)
\]

\[
\Rightarrow \{\Theta^i(x)\} \text{ indeed determine } g_{\mu\nu}(x):
\]

\[
\Rightarrow \text{ The } A^i_\nu(x) \text{ carry all physical (here shape) info!}
\]
How then does $A^\Omega_i(x)$ encode $c^i_{jk}, \omega^i_j, \Omega^i_j$?

- Start with orthonormal frame: $\Theta^i(x) = A^i_j(x)dx^j$

1) How do the $A^i_j(x)$ determine the $c^i_{jk}(x)$?

Recall from Lecture 11:

$$d\Theta^i(x) = -\frac{1}{2} c^i_{jk}(x) \Theta^j(x) \wedge \Theta^k(x)$$

Here:

$$d\Theta^i(x) = A^i_{jk}(x)dx^k \wedge dx^j \quad \text{because of (x)}$$

$$= -\frac{1}{2} c^i_{ab} \Theta^a \wedge \Theta^b = -\frac{1}{2} c^i_{ab} A^a_k A^b_j dx^k \wedge dx^j$$

$$\Rightarrow \quad A^i_{jk} = -\frac{1}{2} c^i_{ab} A^a_k A^b_j$$

$$\Rightarrow \quad c^i_{ab}(x) = -2 A^i_{jk}(x) (A^j(x))'_k (A^k(x))'_b$$

2) The $c^i_{jk}(x)$ yield the $\Gamma^i_{jk}(x)$ through:

$$\Gamma^i_{jk} := \frac{1}{2} \left( c^i_{kj} - g^i_{kj} g^s_{ij} c^s_{kj} - g^i_{ki} g^s_{ij} c^s_{kj} \right)$$

$$+ \frac{1}{2} g^s_{ij} \left( 3 g^i_{kj} - g^s_{ki} g^j_{ij} - g^s_{kj} g^i_{ij} \right)$$

These all vanish because $g^{kj}$ is now orthonormal.

Notice: This simplifies for orthonormal frames with $g^{kj}(x) = g_{ij}$!

3) The $\Gamma^i_{jk}(x)$ yield the $\omega^i_j(x)$:

$$\omega^i_j(x) := \Gamma^i_{kj}(x) \Theta^k(x)$$

4) Recall the 2nd structure equation:

$$\Omega^i_j(x) := dw^i_j + \omega^i_k \wedge w^k_j$$

$$\Rightarrow \quad We \ have: A^i_j \rightarrow \Theta^i \rightarrow c^i_{jk} \rightarrow \Gamma^i_{jk} \rightarrow \omega^i_j \rightarrow \Omega^i_j$$
Recall important identities: (torsionless case)

- **Structure eqm I:**
  \[ \Theta^i = D\Theta^i = d\Theta^i + \omega^i_j \wedge \Theta^j = 0 \]

- **Structure eqm II:**
  \[ \Sigma^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \]
  (Ordinarily: \( \Theta = dx \cdot \Theta \wedge dx \) and \( \omega^i_j = \Gamma^i_{jk} \))

- **Bianchi identity I:**
  \[ \Sigma^i_j \wedge \Theta^j = 0 \]

- **Bianchi identity II:**
  \[ D\Omega^i_j = 0 \]
  (Recall \( R^i_j = \Gamma^i_{jk} + \Gamma^i_{l} \wedge \Gamma^l_{jk} \))

- From diffeomorphism invariance

And, in the case of ON frames:

\[ \omega^\mu_\nu + \omega^\nu_\mu = 0 \]

**Telephic formulation of GR:**

Consider the action, for now, without cosmological constant and without matter:

\[ S'_{\mu\nu} = \frac{1}{16\pi G} \int_B R \sqrt{g} \; d^4x \]

**Recall Hodge \(*\):**

\[ \nu = \frac{1}{p!} \; \nu_i \cdots \nu_p \; \Theta^i \cdots \Theta^p \]

Then:

\[ \ast \nu = \frac{1}{p!} \sqrt{g} \; \epsilon_{i_1 \cdots i_p} \; \nu^{i_1} \cdots \nu^{i_p} \; \Theta^{i_1} \cdots \Theta^{i_p} \]

\[ \ast \nu = \Lambda^0 \rightarrow \Lambda^{n-p} \]

**Thus:**

\[ S'_{\mu\nu} = \frac{1}{16\pi G} \int_B \ast R \; 4\text{-form} \]
Aim now: Re-express $S'_{\mu\nu}$ in terms of $\Omega^{\nu \sigma}$ and $\Omega^{\nu \tau}$. 

Define: "capital $\pi$" is a $(0,2)$ tensor-valued 2-form

$$H_{\alpha \beta} := \star (\pi^{\alpha \beta} \pi^{\sigma \rho}) = \frac{1}{2} \varepsilon_{\rho \sigma \lambda} \pi^{\alpha \beta} \pi^{\sigma \lambda \rho}$$

$$H_{\alpha \beta \gamma} := \star (\pi^{\alpha \beta} \pi^{\sigma \lambda} \pi^{\tau \rho}) = \frac{1}{2} \varepsilon_{\sigma \rho \lambda} \pi^{\alpha \beta} \pi^{\sigma \lambda \rho \tau}$$

to a $(0,3)$ tensor-valued 1-form.

Proposition:

$$\star R = H_{\mu \nu} \wedge \Omega^{\nu \sigma} \quad \text{(4-form)}$$

i.e.: $$\sum' (\pi^{\nu}) = \int H_{\mu \nu} \wedge \Omega^{\nu \sigma}$$

Proof:

Use $\Omega^{\nu \sigma} = \frac{1}{d} R^{\nu \mu} \wedge \theta^\mu \wedge \theta^\lambda \Rightarrow$

$$H_{\mu \nu} \wedge \Omega^{\nu \sigma} = \frac{1}{d} \varepsilon_{\mu \nu \rho \sigma} R^{\rho \mu} \wedge \theta^\delta \wedge \theta^\lambda \wedge \theta^\kappa \wedge \theta^\lambda \wedge \theta^\lambda$$

Use also: $\varepsilon_{\mu \nu \rho} \varepsilon_{\mu \nu \sigma} = 2 (\delta_{\nu \rho} \delta_{\lambda \sigma} - \delta_{\nu \sigma} \delta_{\lambda \rho}) \Rightarrow$

$$H_{\mu \nu} \wedge \Omega^{\nu \sigma} = \frac{1}{d} R^{\nu \mu \sigma} \wedge \theta^\delta \wedge \theta^\lambda \wedge \theta^\kappa \wedge \theta^\lambda \wedge \theta^\lambda = \star R \; \checkmark$$

Proposition: $$DH_{\mu \nu} = 0$$

Recall the first structure equation $DBD^0 = 0$

Proof: $$DH_{\mu \nu} = D(\frac{1}{d} \varepsilon_{\mu \nu \rho \sigma} \pi^\rho \wedge \pi^\sigma) = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} (DB^\rho D\pi^\sigma + D\pi^\rho D\pi^\sigma)$$
The main proposition:

Variation of the action with respect to \( \delta \Theta^{\nu}(x) \) yields:

\[
S(x) = (\delta \Theta^{\nu}) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\sigma} + \delta(\text{something})
\]

It implies:

\[
16\pi G \delta S_{\text{gravity}} = \int_B \delta \Theta^{\nu} \wedge H_{\mu\nu\sigma} \wedge \Omega^{\sigma} + \int_{\partial B} \delta(\text{something})
\]

Definition: The "energy-momentum 1-form" \( T_{\mu} \) is defined as the solution to:

\[
\delta S_{\text{matter}} = \int_B \delta \Theta^{\nu} \wedge (\ast T_{\mu})
\]

\( \Rightarrow \) The equation of motion, i.e., the Einstein equation,

\[
\frac{\delta (S_{\text{grav}} + S_{\text{matter}})}{\delta \Theta^{\nu}} = 0
\]

becomes:

\[
-\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\sigma} = 8\pi G \ast T_{\mu}
\]

Exercise: add the cosmological constant.

Remark: The Einstein form \( G_{\mu} := G_{\mu\nu} \Theta^{\nu} \) obeys

\[
\ast G_{\mu} = -\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\sigma}
\]

\( \Rightarrow \)

\[
G_{\mu} = 8\pi G T_{\mu}
\]
Proof of the main proposition:

\[ S(*R) = (\delta \Theta^\nu) \wedge H_{\mu\nu} \wedge \Omega^\nu + d(\text{something}) \]

Indeed:

\[ S(*R) = (\delta H_{\mu\nu}) \wedge \Omega^\nu + H_{\mu\nu} \wedge \delta \Omega^\nu \]

Consider the first term:

\[ \delta H_{\mu\nu} = \frac{1}{2} \sqrt{|g|} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho \wedge \Theta^\sigma \]

by definition of \( H_{\mu\nu} \) above:

\[ H_{\mu\nu} = \frac{1}{2} \sqrt{|g|} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho \]

\[ \Rightarrow \]

\[ S(*R) = (\delta \Theta^\nu) \wedge H_{\mu\nu} \wedge \Omega^\nu + H_{\mu\nu} \wedge \delta \Omega^\nu \]

Examine this term:

\[ \delta \Omega^\nu = \delta (d \omega^{\nu\rho} + \omega^{\nu\rho} \wedge \omega^\rho) \]

\[ = d \delta \omega^{\nu\rho} + (\delta \omega^\rho) \wedge \omega^\nu + \omega^{\nu\rho} \wedge \delta \omega^\rho \]

\[ \Rightarrow \]

\[ H_{\mu\nu} \wedge \delta \Omega^\nu = d(H_{\mu\nu} \wedge \delta \omega^{\nu\rho}) - (d H_{\mu\nu}) \wedge \delta \omega^{\nu\rho} \]

\[ + H_{\mu\nu} \wedge \delta \omega^{\nu\rho} \wedge \omega^\rho + H_{\mu\nu} \wedge \omega^\rho \wedge \delta \omega^{\nu\rho} \]

by definition of \( \delta \)

\[ = (\delta \omega^{\nu\rho}) \wedge D H_{\mu\nu} + d(H_{\mu\nu} \wedge \delta \omega^{\nu\rho}) \]

by Proposition:

\[ \Rightarrow \]

\[ S(*R) = (\delta \Theta^\nu) \wedge H_{\mu\nu} \wedge \Omega^\nu + d(H_{\mu\nu} \wedge \delta \omega^{\nu\rho}) \]

\[ \checkmark \]
General Relativity as a "gauge theory"

Recall:
\[ \sum_{\nu} \Theta^\nu \Rightarrow \sum_{\nu} H_{\mu\nu} \wedge \Omega^{\mu} \quad \text{Einstein action} \]
\[ -\frac{1}{2} H_{\mu
u} \wedge \Omega^{\nu} = \tilde{\Theta} \rightarrow A^\nu_{\mu}(x) \Theta^\nu(x) \quad \text{Einstein equation} \]

are now the same in all coordinate systems.

In addition:

They are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:
\[ \Theta^\nu(x) \rightarrow \tilde{\Theta}^\nu(x) = A^\nu_{\mu}(x) \Theta^\nu(x) \]
The \( A^\nu_{\mu}(x) \) are local Lorentz transformations.

\[ \uparrow \text{Upshot:} \uparrow \]

We can start with any matter theory that is invariant under global Lorentz transformations and, through general relativity, turn it into a theory that is invariant under local Lorentz transformations.

\[ \Rightarrow \text{Therefore:} \]
Derivatives become covariant derivatives.
A new field is introduced: gravity's \( \Gamma \).

This is analogous to the gauge principle of particle physics:

\[ \Rightarrow \text{A global symmetry is "gauged" to become local.} \]
\[ \Rightarrow \text{Derivatives become covariant derivatives.} \]
\[ \Rightarrow \text{A new field is introduced.} \]
The gauge principle:

Action for a Dirac field (electrons, quarks etc):

\[ S'[\Psi] = \int \bar{\Psi}(ix^\nu \partial_\nu - m) \Psi \, d^4x \]

It has a global symmetry:

\[ \Psi(x) \to \bar{\Psi}(x) = e^{i\alpha} \Psi(x), \text{ i.e., } \bar{\Psi}(x) = e^{-i\alpha} \bar{\Psi}(x) \]

\[ \Rightarrow S[\Psi] \to S[\bar{\Psi}] = S'[\Psi] \]

However, no local symmetry:

\[ \Psi(x) \to \bar{\Psi}(x) = e^{i\alpha(x)} \Psi(x), \bar{\Psi}(x) \to \bar{\Psi}(x) = e^{-i\alpha(x)} \bar{\Psi}(x) \]

\[ S'[\Psi] \to S[\bar{\Psi}] \neq S'[\Psi] ! \]

Gauge principle: Introduce a new field \( A_\mu(x) \) that transforms so as to absorb the extra term:

\[ S'[\Psi, A] := \int \bar{\Psi}(x) \left( i x^\nu (\partial_\nu + i A_\nu(x)) - m \right) \Psi(x) \, d^4x \]

"covariant derivative"

Now under

\[ \Psi(x) \to \bar{\Psi}(x) = e^{i\alpha(x)} \Psi(x) \]

\[ A_\mu(x) \to \bar{A}_\mu(x) := A_\mu(x) - i \partial_\mu \alpha(x) \]

the action obeys:

\[ S'[\Psi, A] \to S'[\bar{\Psi}, \bar{A}] \]

\[ = \int \bar{\Psi}(x) e^{-i\alpha(x)} \left( i x^\nu (\partial_\nu + i A_\nu - i \partial_\nu \alpha(x) - m) \right) e^{i\alpha(x)} \Psi(x) \, d^4x \]

\[ = S'[\Psi, A] \]
Generalization to Yang-Mills theory

Gauging \( Y(x) \rightarrow e^{i \xi(x)} Y(x) \) introduces \( A_\mu(x) \).

and \( A_\mu(x) \) turns out to exist. The EM 4-potential.

We "derived" the electromagnetic force!

Notice: \( e^{i \xi(x)} \in U(1) \)

\[ U(1) = \left\{ G \in \mathbb{C} \mid G^+ = G^{\ast} \right\} \]

Now give the Dirac particle an extra index (isospin bundle)

\[ S'[\Psi] = \int \overline{\Psi}_a \left( i \gamma^\mu \left( \delta_{ab} \partial_\mu + i B_\mu \right) \right) \Psi_b \ d^4x \]

It's invariant under:

\[ \Psi_a(x) \rightarrow G_{ab} \Psi_b(x) \]

where \( G \in SU(N) \)

\[ SU(N) = \left\{ G \in M_n(\mathbb{C}) \mid G^+ = G^{\ast} \right\} \]

Now, we gauge, i.e., require invariance under:

\[ \Psi_a(x) \rightarrow G_{ab} \Psi_b(x) \quad \text{where} \quad G \in SU(N) \]

Invariance of the action now requires new field \( B_\mu(x) \):

\[ S'[\Psi] = \int \overline{\Psi}_a \left( i \gamma^\mu \left( \delta_{ab} \partial_\mu + i B_\mu \right) \right) \Psi_b \ d^4x \]

"covariant derivative"

and \( B_\mu(x) \rightarrow \tilde{B}_\mu(x) \): complicated

Here: \( T_{ab} \in su(N) \) are Lie algebra basis, i.e. they are generators of infinitesimal \( SU(N) \) transformations.

Upshot:

- \( N=2 \) Weak force (though mixed with \( N=1 \) EM)
- \( N=3 \) Strong force (QCD)
Recall:

$$\sum_{\mu\nu} (\Theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\nu}$$  
Einstein action

$$-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \ast T_{\mu}$$  
Einstein equation

are the same also with any choice of ON bases in
the tangent spaces, i.e., we have a local symmetry under:

$$\Theta^\nu(x) \to \tilde{\Theta}^\nu(x) = A^\mu_{\nu}(x) \Theta^\nu(x)$$

The $A^\mu_{\nu}(x)$ are local Lorentz transformations.

Our covariant derivative:

$$\nabla_{\nu} (\nabla^\nu(x) e_{\nu}) = \left( \frac{\partial}{\partial x^\nu} \nabla^\nu(x) e_{\nu} + \nabla^\nu(x) \omega^\nu_{\sigma}(e_{\nu}) e_{\sigma} \right)$$

Do the $\omega^\nu_{\sigma}$ indeed generate infinitesimal Lorentz transformations? It is now gravity!

Interpretation of the connection in ON frames:

Q: The connection 1-forms $\omega^\nu_{\sigma}$ are not, we know,
tensor-valued 1-forms. Wherin do they take their values?

A: The connection 1-forms take values in the set of
infinitesimal Lorentz transformations

Intuition?

The connection yields the change under infinitesimal
parallel transport - and parallel transport preserves
the metric, i.e. it preserves the lengths of vectors, i.e.
the change can only be an infinitesimal "rotation", i.e.
an infinitesimal Lorentz transformation.
Recall: "Lorentz transformations $\Lambda^\nu_\mu$ are linear maps obeying:

$$\Lambda^\nu_\alpha \Lambda^\alpha_\beta \eta_{\mu\nu} = \eta_{\mu\beta}$$

$\Rightarrow$ Infinitesimal Lorentz transformations

$$\Lambda^\nu_\mu = \delta^\nu_\mu + \varepsilon^\nu_\mu$$

with $$(\varepsilon^\nu_\mu)^2 = 0$$

obey:

$$(\delta^\nu_\alpha + \varepsilon^\nu_\alpha)(\delta^\nu_\beta + \varepsilon^\nu_\beta) \eta_{\mu\nu} = \eta_{\mu\beta}$$

i.e.:

$$\varepsilon^\nu_\alpha \eta_{\mu\beta} + \varepsilon^\nu_\beta \eta_{\alpha\mu} = 0$$

$\Rightarrow$ Infinitesimal Lorentz transformations "JLT" are given by

all $\Lambda^\nu_\mu = \delta^\nu_\mu + \varepsilon^\nu_\mu$ which obey:

$$\varepsilon^\mu_\beta + \varepsilon^\beta_\mu = 0$$

Q: Are connection 1-forms JLT valued?

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivation)

$$\text{Dt}^\nu_{abcd} = \text{d}^\nu_{abcd} + \omega_{ia} \text{e}^a_{ibcd} + \ldots - \omega_{i} \text{e}^{i}_{abcd} - \ldots$$

Thus:

Recall that by using a tetrad, one observes that $\omega_{\mu\nu} = \Omega_{\mu\nu}$ everywhere.

$$0 = \nabla g_{\mu\nu} = Dg_{\mu\nu} = dg_{\mu\nu} - \omega_{\mu} \wedge g_{\nu} - \omega_{\nu} \wedge g_{\mu}$$

i.e. $0 = \omega_{\mu\nu} + \omega_{\nu\mu}$.