Friedmann-Lemaître cosmological solutions

Experimental evidence:

- The universe is and has been expanding.
- It appears to be spatially essentially isotropic and homogeneous on scales larger than a few (3-4) billion light years.
  
  \[ \text{(see e.g. Sloan Digital Sky Survey (SDSS) at www.sdss.org)} \]

Idealizing models:

- Assume perfect spatial isotropy and homogeneity:
  
  \[ \Rightarrow \text{"Friedmann & Lemaître" (later Robertson & Walker) spacetimes} \]

Concretely:

We assume we can model spacetime as a manifold \((M, g)\)

with:

\[
M = J \times \Sigma
\]

\[
g = -dt^2 + a^2(t) g_\Sigma
\]

Here:

- \(J\) is an interval, \(J \subset \mathbb{R}\), and \(t \in J\) is called "cosmic time".
- \(a(t)\) is called the "scale factor".

\((\Sigma, g)\) is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of homogeneity at each point in cosmic time.
What are the possible Riemannian manifolds of constant curvature?

The Riemann tensor $\bar{R}^{\text{sym}}_{\text{sym}}$ must be expressible in terms of a constant, say $K$, which fixes the curvature's strength, and the tensorial part can only depend on the metric $\bar{g}$.

Given the index symmetries of $\bar{R}^{\text{sym}}_{\text{sym}}$, it should (and does) take the form:

$$\bar{R}^{\text{sym}}_{\text{sym}} = K \left( \bar{g}^{ij} \bar{g}^{je} - \bar{g}^{ie} \bar{g}^{je} \right) \quad (\star)$$

$$\Rightarrow \bar{R}^{\text{sym}}_{\text{sym}} = 2K \bar{g}^{je}, \quad \bar{R} = 6K$$

Using a "Tried" form:

$$\bar{R}^{\text{sym}}_{\text{sym}} \equiv \frac{1}{2} \bar{R}^{\text{sym}}_{\text{sym}} \bar{\theta}_i \wedge \bar{\theta}_j \wedge \bar{\theta}_k \wedge \bar{\theta}_l$$

(over basis of $T_p(\Sigma), V_P$)

Role of the signature of $K$:

$K > 0$: $\Rightarrow \Sigma$ is a 3-dim. sphere (that can be embedded e.g. in a 4-dim euclidean (i.e. flat) space: closed universe

$K = 0$: $\Rightarrow \Sigma$ is euclidean $\mathbb{R}^3$. flat; infinite universe

$K < 0$: $\Rightarrow \Sigma$ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These $\Sigma$ also possess $\infty$ volume.

Note: $\bar{R}$ and therefore $K$ have units $\frac{1}{\text{(length)}^2}$. Thus, by suitable choice of unit of length, we can choose units of length so that: (This is usually done in cosmology)

$K = -1, 0, 1$
A tetrad for spacetime: \[ g = -dt^2 + a^2(t)\hat{g} \]

- Define a convenient tetrad, i.e., ON basis of each \( T_p(M) \):
  \[ \Theta^0 := dt \quad \text{with } t = \text{cosmic time of above} \]
  \[ \Theta^i := a(t) \bar{\Theta}^i \quad \text{with } \bar{\Theta}^i \text{ being the dual of } \Sigma \]

- Note: The \( \bar{\Theta}^i \) were chosen ON with respect to \( g \).
  The \( \Theta^i \) are ON with respect to \( \hat{g} \).

We then have, e.g.:

- 1st structure equation on \( \Sigma \): \( \mathcal{J}^{ij} = (1,2,3) \)
  \[ d\bar{\Theta}^i + \bar{\omega}^i_j \wedge \Theta^j = 0 \] (\( \Sigma 1 \))

  Recall: The Cartan structure equations express the torsion and curvature.
  Form in terms of the connections from.

- 1st structure equation on \( M \): \( \mathcal{J}^{ij} = (\mu, \nu = 0,1,2,3) \)
  \[ d\Theta^i + \omega^i_j \wedge \Theta^j = 0 \] (M 1)

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**Determine the 4-connection \( \omega^i_j \), \( \text{in spatially isotropic & homogeneous case} \)**

**Strategy:** Calculate \( d\Theta^i \) in two ways:

1. \[ d\Theta^i = d(a \bar{\Theta}^i) = (da) \wedge \bar{\Theta}^i + a \, d\bar{\Theta}^i \]

   \[ = \left(\frac{da}{dt} \right) dt \wedge \bar{\Theta}^i - a \bar{\omega}^i_j \wedge \Theta^j \] (use \( a \bar{\Theta}^i = \Theta^i \))

   \[ = a \, \Theta^i \wedge \bar{\Theta}^i - \bar{\omega}^i_j \wedge \Theta^j \]

   (use \( \Theta^i = \bar{\Theta}^i \))

   \[ = a \, \Theta^i \wedge \bar{\Theta}^i - \bar{\omega}^i_j \wedge \Theta^j \]

   (use \( \Theta^i \wedge \bar{\Theta}^i = \Theta^i \wedge \Theta^i \))

   \[ = a \, \Theta^i \wedge \Theta^i - \bar{\omega}^i_j \wedge \Theta^j \]

   \[ (A) \]

2. \[ d\Theta^i = -\omega^i_o \wedge \Theta^o = -\omega^i_o \wedge \Theta^0 - \omega^i_j \wedge \Theta^j \] (B)

**Compare eqns A&B \Rightarrow**

\[ \omega^i_o = \frac{a}{2} \Theta^i \quad \text{and} \quad \omega^i_j = \bar{\omega}^i_j \]

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\( \text{Instructions: Compare connection in variational & affine frame} \)
The curvature 2-form:

Recall: 2nd structure equations:

\[ \Omega^i_{\phantom{i}j} = d\omega^i_{\phantom{i}j} + \omega^i_{\phantom{i}k} \wedge \omega^k_{\phantom{k}j} \quad (M2) \]

\[ \overline{\Omega}^i_{\phantom{i}j} = d\overline{\omega}^i_{\phantom{i}j} + \overline{\omega}^i_{\phantom{i}k} \wedge \overline{\omega}^k_{\phantom{k}j} \quad (\Sigma 2) \]

for \( i, j \in \{1, 2, 3\} \) (afterwards we will calculate \( \Omega^i_{\phantom{i}j}, \overline{\Omega}^i_{\phantom{i}j} \))

\[ \Rightarrow \Omega^i_{\phantom{i}j} = d\overline{\omega}^i_{\phantom{i}j} + \overline{\omega}^i_{\phantom{i}k} \wedge \overline{\omega}^k_{\phantom{k}j} \quad \text{use (Box)} \Rightarrow \]

\[ = d\overline{\omega}^i_{\phantom{i}j} + \overline{\omega}^i_{\phantom{i}k} \wedge \overline{\omega}^k_{\phantom{k}j} + \omega^i_{\phantom{i}k} \wedge \omega^k_{\phantom{k}j} \]

\[ \Rightarrow \overline{\Omega}^i_{\phantom{i}j} = \omega^i_{\phantom{i}k} \wedge \omega^k_{\phantom{k}j} \]

Recall also:

\[ \overline{\Omega}^i_{\phantom{i}j} = \kappa \overline{\theta}^i \wedge \overline{\theta}^j = \frac{\kappa}{a^2} \theta^i \Lambda \theta^j \quad \text{(It was a consequence of spatial isotropy & homogeneity)} \]

\[ \Rightarrow \quad \Omega^i_{\phantom{i}j} = \frac{\kappa}{a^2} \theta^i \Lambda \theta^j + \frac{a^2}{a^2} \theta^i \Lambda \theta^j \quad \text{Recall from equations (Box)}: \]

\[ \dot{\omega}^i = \frac{1}{2} \theta^i, \quad \omega^i_{\phantom{i}0} = -\frac{1}{2} \theta^i \]

\[ \omega^i_{\phantom{i}0} = \frac{1}{2} \theta^i, \quad \omega^i_{\phantom{i}0} = \frac{1}{2} \theta^i \]

\[ \Rightarrow \quad \Omega^i_{\phantom{i}j} = \frac{\kappa + a^2}{a^2} \theta^i \Lambda \theta^j \]

Similarly, one calculates:

Exercise: check

\[ \Omega^0_{\phantom{0}i} = \frac{a}{a} \theta^0 \Lambda \theta^i \]

Calculate the Einstein tensor:

Recall:

\[ \Omega_{\mu\nu} = \frac{1}{a} R_{\mu\nu\sigma\tau} \theta^\sigma \wedge \theta^\tau \]

\[ \Rightarrow \text{We can read off } R_{\mu\nu\sigma\tau}. \]
We obtain the Ricci tensor $R_{\mu
u}$ and the curvature scalar $R$.

We obtain the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

**Result:**

$$G_{00} = 3\left(\frac{a^2}{a^2} + \frac{\kappa}{a^2}\right)$$

$$G_{ii} = -2\frac{\dot{a}}{a} - \frac{a^3}{a^2} - \frac{\kappa}{a^2}$$

$$G_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

i.e., $G_{\mu\nu}$ is diagonal in this frame.

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The energy-momentum tensor:

- From $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we obtain that $T_{\mu\nu}$ must also be diagonal.

- Recall the interpretation of the entries of a diagonal $T_{\mu\nu}$ in terms of matter energy density $\rho$, matter pressure $p$ and cosmological constant $\Lambda$ at the origin of geodesic coordinates:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & -\frac{1}{3} \rho \Lambda \end{pmatrix}$$

Why this factor here? (Because $\Lambda$ is a correction)

The only nontrivial dynamics of matter is here its equation of state:

$$\rho = \rho(p) \text{ and } p = p(\rho)$$
What kind of matter causes such a $T_{\mu\nu}$?

Proposition:

The $T_{\mu\nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* The matter doesn't have to be fluid - it could also be e.g. a suitable quantum field.
* But the high symmetry of a F.L. spacetime requires that the matter's $T_{\mu\nu}$ matches that of a perfect fluid.

Proof: Consider the 4-vector field dual to $\Theta^\alpha$:

\[ u = \frac{\Theta}{\nabla} = e_0 \text{, i.e.: } u = u^\mu e_\mu \text{ with } u^0 = 1, \ u^i = 0. \]

Using $u$, $T_{\mu\nu}$ takes the form that characterizes a perfect fluid:

\[ T_{\mu\nu} = (\rho + p) u^\mu u^\nu + (\rho - \Lambda) g_{\mu\nu} \]

Q: If the matter is a fluid, what's the vector field $u$?

Recall:

\[ \omega^0 = \frac{1}{2} \dot{r} \]
\[ \omega^i = \frac{1}{2} \dot{\theta} \]
\[ \omega^\phi = \frac{1}{2} \dot{\phi} \]
\[ \omega^* = 0 \]

A: We are a particle of the fluid and $u$ is our velocity:

Why? $u$ is tangent to timelike geodesics (that stand still in space) because $u^\mu \nabla_\mu u_\nu = 0$.

\[ \nabla_\mu u = \nabla_\mu e_0 = \omega^0_\mu e_\mu = \frac{a}{a} \delta^0_\mu e_\mu = 0 \]

\( \mu = 0,1,2,3 \)
The Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

now consists of:

$$G_{00} = 8\pi G T_{00} \Rightarrow 3 \left( \frac{a^2}{a^2} + \frac{\kappa}{a^2} \right) = 8\pi G \rho + \Lambda$$ \hspace{1cm} \text{“Friedmann equation” (A)}

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow -2 \frac{\ddot{a}}{a} - \frac{a^2}{a^2} - \frac{\kappa}{a^2} = 8\pi G p - \Lambda$$ \hspace{1cm} \text{(B)}

Notice that $\Lambda$ contributes

- a positively to the energy but
- a negatively to the pressure.

Observation: $k/a^2$ occurs in (A) and (B), i.e., we can eliminate it:

$$-\frac{1}{a} \left( \text{Eqn}(B) + \frac{1}{3} \text{Eqn}(A) \right) \text{ yields:}$$

$$\ddot{a} = -\frac{1}{2} a \left( 8\pi G \left( \frac{\rho}{3} + p \right) - \frac{1}{2} a \Lambda \left( -1 + \frac{1}{3} \right) \right)$$

$$\Rightarrow \ddot{a} = -\frac{4\pi G a}{3} (\rho + 3p) + \frac{1}{3} a \Lambda$$

Thus for all $k$: For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant $\Lambda$ can make $\ddot{a} > 0$. 
Experimental evidence?

- Supernova distance versus brightness data and evidence from cosmic background radiation:
  \[ \frac{\alpha}{\beta} \approx 0 \text{ now}! \]

\[ \Rightarrow \text{At present, energy is already sufficiently diluted so that } \Lambda \text{ dominates over } g : \approx 70\% \, \Lambda \text{ and } \approx 30\% \, g \text{ (dark matter)} \]

\[ \Rightarrow \text{In the far future, } g \text{ & } p \text{ will have diluted } \to 0, \]

\[ (\text{If something other than } \Lambda \text{ will dominate then experiments indicate that indeed a faster than exponential expansion may be under way. This cannot result from just } \Lambda \text{ dominance alone.)} \]

\begin{align*}
\text{Solutions:} & \\
3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) &= \Lambda \\
&
\begin{cases}
\cosh(\frac{\dot{a} \tau}{\sqrt{3}}) & \text{for } k = 1 \\
\exp(\frac{\dot{a} \tau}{\sqrt{k}}) & \text{for } k = 0 \\
\sinh(\frac{\dot{a} \tau}{\sqrt{3}}) & \text{for } k = -1
\end{cases}
\end{align*}

\[ \Rightarrow \text{Exponential expansion is predicted!} \]

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General solution strategy with cosm. constant and matter:

- We have 3 unknown functions of time:
  \[ a(\tau), \, g(\tau), \, p(\tau) \]

and we have 3 equations that they obey:

\begin{align*}
\text{Eqs. A, B and an equation of state } p = p(g) \text{ that depends on the "matter":} & \\
\end{align*}

\[ p_n(\tau) = -\frac{\dot{a}}{a} \text{ for pure vacuum energy (e.g., in very early universe)} \]
\[ p(\tau) = \frac{1}{3} g \text{ for pure radiation (e.g., in the early universe)} \]
\[ p(\tau) = 0 \text{ for pure dust (e.g., middle age universe, approaching a flat } \Lambda \text{ dominated one)} \]

- Observation:
  \[ (\text{Eqs. A}) \]
  \[ \text{The Friedmann eqns. only contains } a, g \text{ but not } p! \]
Idea:

- Try to express $\tilde{g}$ as a function of $a$ to obtain $\tilde{g} = \tilde{g}(a)$.
- Using $\tilde{g}(a)$, the Friedmann eqn becomes an ordinary differential equation only for $a(t)$ and we are done!

Indeed, a key equation helps us to find $\tilde{g}(a)$:

**Proposition:** The Einstein eqns $\Lambda, \beta$, i.e., $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, imply:

\[
\frac{d}{da}\left(\frac{\tilde{g}}{a^3}\right) = -3\rho a^2
\]

(P)

Indeed, when the parameter $\Lambda \equiv w\tilde{g}$ is known, (P) yields $\tilde{g}(a)$:

- For dust, $\rho = 0$ \(\Rightarrow\) $\tilde{g} \sim a^{-3}$
- For radiation, $\rho = \beta/3$ \(\Rightarrow\) $\tilde{g} \sim a^{-4}$
- For pure $\Lambda$: $\rho = -\Lambda$ \(\Rightarrow\) $\tilde{g} = \text{const}$ \(\Rightarrow\) $\tilde{g}$ of vacuum energy does not dilate!

**Intuitive meaning of (P)**

(P) in the GR version of the continuity equation for dust without heat flow

- Adiabatic expansion: \(dE = -\rho dV\)
- With $V = a^3$, $E = \tilde{g}V$ it yields:
  \[
  d(a^3\tilde{g}) = -\rho d(a^3) = -3\rho a^2 da
  \]
- Thus: \(\frac{d}{da}(a^3\tilde{g}) = -3\rho a^2\) which is indeed (P).
Exact proof of proposition (P):

1. The Einstein equation $\mathcal{G}^{\mu\nu} = 8\pi G T^{\mu\nu}$ and $\mathcal{G}^{\mu\nu};\nu = 0$ imply $T^{\mu\nu};\nu = 0$

2. Here: $T^{\mu\nu} = (\mathfrak{S} + \rho) u^\nu u^\mu + \rho g^{\mu\nu}$

Thus:

$$0 = T^{\mu\nu};\nu = (\mathfrak{S};\nu + p;\nu) u^\mu u^\nu + (\mathfrak{S} + p) u^\mu u^\nu;\nu + p;\nu g^{\mu\nu}$$

(using $u^\mu u_\mu = 1$)

$$= (\mathfrak{S};\nu + \mathfrak{D} u^\nu) u^\mu + (\mathfrak{S} + p) u^\nu;\mu u^\nu + p;\mu u^\nu$$

$$= -\mathfrak{D} u^\mu - \mathfrak{D} p - (\mathfrak{S} + p) \mathfrak{D} u + u^\mu p;\mu$$

$$\Rightarrow 0 = \mathfrak{D} u^\mu + (\mathfrak{S} + p)(\mathfrak{D} u)$$

(X)

It remains now to calculate $\mathfrak{D} u$:

Recall that

$$\mathfrak{D} u^\mu = \mathfrak{D} (u^\mu) = \Theta^\mu_\nu \mathfrak{D} e^\nu$$

$$\Rightarrow \mathfrak{D} u = \Theta^\mu_\nu \mathfrak{D} e^\nu = \Theta^\mu_\nu (\mathfrak{D} e^\nu)$$

$$= \Theta^\mu_\nu \left( \omega^\nu_\alpha (e_\alpha) e^\nu \right) = \omega^\nu_\alpha (e_\alpha) = \omega^\nu_\alpha (e_\alpha)$$

$$\Rightarrow \text{Eqn. (X) becomes:}$$

$$\mathfrak{D} u^\mu + (\mathfrak{S} + p)(\mathfrak{D} u) = 0$$

Recall: $u^\mu = \frac{d\rho}{d\xi}$

Thus:

$$\frac{d\rho}{d\xi} + \left( \frac{3a^3}{a} (\mathfrak{S} + p) \right) = 0$$

$$\Rightarrow \frac{d\rho}{d\xi} + \frac{3a^3}{a} (\mathfrak{S} + p) a^2 = 0$$

Thus:

$$\frac{d\rho}{d\xi} a^3 + 3(\mathfrak{S} + p) a^2 = 0$$

$$\frac{ds}{dx} \frac{dt}{da} a^3 + 3(\mathfrak{S} + p) a^2 = -3pa^2$$

$$\Rightarrow \frac{ds}{da} a^3 + 53a^2 = -3pa^2$$

$$\Rightarrow \frac{d}{da} (\frac{d\rho}{ds}) = -3pa^2$$

This is Eqn (P) \(\checkmark\)