Horizons & Singularities

Local causal structure

The metric, g, not only defines the "shape" of a pseudo-Riemannian manifold, it also defines what is causal and what is acausal: (by defining what is space, null, time)

Preparation: Consider an arbitrary point \( p \in M \) and an arbitrary "convex normal neighborhood" of \( p \), i.e., a set \( U \subset M \) with \( p \in U \) for which holds:

\[ g_{ij} r^i r^j > 0 \text{ for } r \in U \Rightarrow \text{there exist a unique geodesic connecting } q \text{ and } r. \]

**Lemma:** There always exists such a neighborhood.

Now consider in \( U \):

- Past light cone is determined by the metric.
- Some arbitrary coordinate system in which the hyperplanes t = constant are space-like.

**Definition:** In order for the laws of matter fields \( \Phi \) to be called "locally causal" (and therefore reasonable), their equations of motion must allow one to calculate \( \Phi(p) \) from only the values \( \Phi(q) \) and finite order derivatives \( \Phi(q) \), ..., for all \( q \in S \).
Remark: In Newton's theory these data don't suffice, because there: \( c = \infty \)

Equivalently: The laws of matter fields are locally causal if signals can be sent between events \( q, p \in \mathcal{U} \) only if there is a curve \( \gamma \in \mathcal{U} \) with \( \gamma(t_1) = q, \gamma(t_2) = p \) whose tangents are non-spacelike:

\[
g(\gamma(t), \gamma(t)) \leq 0 \text{ for all } t \in [t_0, t_1]
\]

Question: Assume that on a differentiable manifold \( M \) only a causal structure is given. To what extent fixes this \( g \)?

Answer: Nearly completely!

Theorem:

Assume that on a differentiable manifold \( M \) we don't know the metric, i.e., we can't evaluate \( g(\xi, \eta) \)

but assume that for all \( p \in M \) and all \( \xi, \eta \in T_p(M) \) we know for each \( \xi \) whether it is space-, light- or time-like, i.e. assume we know:

\[
\operatorname{sign}(g(\xi, \xi)) \text{ for all } p \in M, \xi \in T_p(M)
\]

Then, this information already determines the metric tensor up to conformal transformations, i.e., we obtain:

\[
f(\xi) g_{\mu \nu}(x)
\]

for unspecified scale function: "conformal factor" also called "holographic frame"
Remark:

Conformal transformations affect only the length of vectors but leave their mutual "angles" invariant:

\[
\cos(\theta(\xi, \eta)) = \frac{\xi(x, y)}{g(\xi(x, y))} \bigg|_{\xi(x, y) = \eta(x, y)}
\]

Proof: Consider a timelike \( \xi \) and a spacelike \( \eta \).

Are there linear combinations

\[
\xi + \lambda \eta
\]

that are light-like? If yes, we can assume that we know these \( \lambda \) from knowing the causal structure!

\[\begin{array}{c}
\text{Need to solve this quadratic equation in } \lambda: \\
f(\lambda) = g(\xi + \lambda \eta, \xi + \lambda \eta) = 0 \quad (4)
\end{array}\]

\( i.e.: g(\xi + \lambda \eta, \xi + \lambda \eta) = 0 \)

Eq. (4) has two roots \( \lambda_1, \lambda_2 \). Are they real?

Yes, because:

- \( \xi \) timelike \( \Rightarrow f(0) < 0 \)

- \( \eta \) spacelike \( \Rightarrow f(\lambda) > 0 \) for large enough \( \lambda \)

(1) \( f(0) = 0 \) has one real root

(2) Both roots, \( \lambda_1, \lambda_2 \), of \( f(\lambda) = 0 \) are real.

Since by assumption we can identify all null vectors we can assume \( \lambda_1, \lambda_2 \) known.
Lemma:

\[ \frac{g(\eta,\eta)}{g(\xi,\xi)} = \lambda, \lambda' \]

Thus, the ratio \( \frac{g(\xi,\xi)}{g(\eta,\eta)} \) can be assumed known for all timelike \( \xi \) and all spacelike \( \eta \).

Proof: From \( g(\xi + \lambda_1 \eta, \xi + \lambda_2 \eta) = 0 \)

we have: \( g(\xi,\xi) + 2 \lambda_1 g(\xi,\eta) + \lambda_1^2 g(\eta,\eta) = 0 \)

and: \( g(\xi,\xi) + 2 \lambda_2 g(\xi,\eta) + \lambda_2^2 g(\eta,\eta) = 0 \)

Eliminate \( g(\xi,\eta) \Rightarrow \frac{g(\xi,\xi)}{g(\eta,\eta)} = \lambda, \lambda' \)

Exercise: Show this.

Corollary:

Also, the ratios \( \frac{g(\xi,\xi)}{g(\xi',\xi')} \) for \( \xi, \xi' \) both timelike

(or both spacelike) can be assumed known:

\[ \frac{g(\xi,\xi)}{g(\eta,\eta)} = \lambda, \lambda' \Rightarrow \frac{g(\xi,\xi')}{g(\eta,\eta)} = \frac{\lambda}{\lambda'} \cdot \frac{\lambda'}{\lambda} = \frac{\lambda^2}{\lambda^2} \]

Corollary:

Consider arbitrary non-null vectors \( \alpha, \beta \).

Then

\[ g(\alpha,\beta) = \frac{1}{2} \left[ g(\alpha,\alpha) + g(\beta,\beta) - g(\alpha + \beta, \alpha + \beta) \right] \]

and thus: By Lemma, all these ratios can be assumed known.
Conclusion:

Therefore, if it is known which vectors are timelike, spacelike or null, then it is possible to calculate

\[ g(d, b) \text{ at all } p \in M \text{ for all } d, b \in T_p(M) \]

up to a scalar prefactor. \( \Rightarrow \) Proof of Theorem complete.

Interpretation:

The causal structure alone already determines:

- the "angles" between vectors
- the "lengths" of vectors up to a positive scalar function.

An application to QFT: arXiv:1510.02725 w. prev. students of this course!

Implications:

Space times \((M, g)\) and \((M, \tilde{g})\) for which

\[ \tilde{g} = \phi g \]

where \(\phi\) is a change of metric

possess the same causal structure.

\( \Rightarrow \) Space times fall into "conformal equivalence classes" within which the local causal structure is invariant.

This is very useful to help intuition:

Choose a conformally equivalent spacetime, for which space and time are conformally so much squeezed that infinities turn into a finite distance, all while \(45^\circ\) remain \(45^\circ\) degrees like conformity.
Application: Penrose diagrams

Example: Consider Minkowski space, \((M, g)\) in spherical coordinates:

\[
g = -dx^0 \otimes dx^0 + dx^r \otimes dx^r + dx^\theta \otimes dx^\theta + dx^\phi \otimes dx^\phi
\]

will \(-\infty < t < \infty, 0 \leq r < \infty, 0 \leq \theta < \pi, 0 \leq \phi < 2\pi\)

Now consider the spacetime \((\tilde{M}, \tilde{g})\) given by:

\[
\tilde{g} = dt \otimes dt + dr \otimes dr + \sin^2(r^2)(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)
\]

with \(-\pi < \tilde{t} + \tilde{r} < \pi, -\pi < \tilde{t} - \tilde{r} < \pi, \tilde{r} > 0, 0 \leq \tilde{\theta} < \pi, 0 \leq \tilde{\phi} < 2\pi\)

The spacetimes \((M, g), (\tilde{M}, \tilde{g})\) are related by a diffeomorphism \(\tilde{M} \rightarrow M:\)

\[
t = \frac{1}{2} \tan\left(\frac{1}{2} (\tilde{t} + \tilde{r})\right) + \frac{1}{2} \tan\left(\frac{1}{2} (\tilde{t} - \tilde{r})\right)
\]

\[
r = \frac{1}{2} \tan\left(\frac{1}{2} (\tilde{t} + \tilde{r})\right) - \frac{1}{2} \tan\left(\frac{1}{2} (\tilde{t} - \tilde{r})\right)
\]

The diffeomorphism is not isometric, but it is conformal:

\[
g_{\mu\nu} = \phi \tilde{g}_{\mu\nu} \text{ with } \phi = \frac{1}{4} \sec^2\left(\frac{1}{2} (\tilde{t} + \tilde{r})\right) \sec^2\left(\frac{1}{2} (\tilde{t} - \tilde{r})\right)
\]

Thus, \((M, g)\) and \((\tilde{M}, \tilde{g})\) have the same causal structure, although \(-\pi < \tilde{t} + \tilde{r} < \pi\) and \(-\pi < \tilde{t} - \tilde{r} < \pi\) and \(\tilde{r} > 0\).

**Legend:**

- **Continuous (green) lines:** Infinities
- **Dotted (green) line:** Radial line
- **Singularity (red):** double line.

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\[\tilde{t}\] is future time-like infinity (chronological past is red)

\[\tilde{r}^{-}\] "Semi-plan": future light-like to \(\tilde{t}^{-}\) light-like

\[\tilde{r}^{+}\] "Semi-plan": future light-like to \(\tilde{t}^{+}\) light-like

\[\tilde{r}^=\] "Semi-plan": future light-like to \(\tilde{t}^=\) light-like

all events of Minkowski space are in here

\[\tilde{t}^=\] is spacelike infinity

\[\tilde{r}^=\] is spacelike infinity
Examples:

A) geodesic, massive observer, sitting at $\tau_1$.

B) same but then uniformly accelerating.

C) light ray

Definition:

An observer's Event horizon (if any) in the boundary of the past of this observer's future causal infinity.

i.e., the event horizon in the boundary of the set of those events that can possibly ever influence the observer, i.e., it's the boundary of the set of events the observer can ever learn about.
Recall FL metric for K=0 (i.e. spatially flat $\mathbb{R}^3$)

\[ g^{(FL)} = -dt \otimes dt + a^2(t)dx \otimes dx + a^2(t)r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \]

Change to a new time variable $\tau$: "Conformal time"

\[ \tau(t) = \int_{t_0}^{t} \frac{1}{a(t')} dt' \]

Why useful? Notice:

\[ \frac{d\tau}{dt} = \frac{1}{a(t)} \Rightarrow dt = a(t) d\tau \]

\[ g^{(FL)} = a^2(t) \left( -d\tau \otimes d\tau + a^2(t)r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \right) \]

\[ g^{(FL)} = a^2(t) \gamma \]

\[ g^{(FL)} \text{ is conformally equivalent to Minkowski space!} \]

\[ \Rightarrow \text{can re-use some Penrose diagram, except range of time } \tau \text{ may be smaller!} \]

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Penrose diagrams of FL. cosmologies: (with K=0)

**Example:** Radiation dominated universe: \( a(t) = \frac{t}{t_0} \), \( t > 0 \)

\[ \tau(t) = \int_{t_0}^{t} \frac{1}{\sqrt{t'}} dt' = 2 \sqrt{t} \bigg|_{t_0}^{t} = 2t \Rightarrow \tau > 0 \]

\[ \Rightarrow \text{Obtain } (\tau, r) \text{ diagram with } \tau > 0: \]

\[ \begin{align*}
\tau = 0 & \quad \text{Big bang singularity} \\
\text{\( \mathcal{I}^+ \) (future infinity)} & \quad \text{\( \mathcal{I}^- \) (past infinity)} \\
\text{\( \mathcal{I}^+ \) "Scri plus" } (t = +\infty) & \quad \text{\( \mathcal{I}^- \) "Scri minus" } (t = 0) \\
r = 0 & \quad \text{Particle horizon of } \theta \\
(\text{space-like infinity}) & \end{align*} \]

**Notice:** Singularity at \( t = 0 \) assumed. (Some FL geodesics are unbound, e.g. do not exist at \( t = 0 \))

At finite \( t \), an observer can see only a finite distance.

**Def:** This distance is called the observer's "Particle Horizon" at time \( t \).
Particle horizon:

How far away, \( d_p \), is the particle horizon at time \( t^2 \)?

Recall:

\[
g = -dt^2 + a^2(t) \, d\theta^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2)
\]

Consider a light ray \( \gamma(t) = (\gamma^0(t), \gamma^1(t), 0, 0) \), i.e., emitted radially.

Its tangent is null \( \gamma^0(t) \frac{d\gamma^0}{dt} - a^2(t) \frac{d\gamma^1}{dt} = 0 \), i.e.,

\[
\left( \frac{d\gamma^0(t)}{dt} \right)^2 - a^2(t) \left( \frac{d\gamma^1(t)}{dt} \right)^2 = 0
\]

Note: \( \gamma^0(t) = t(t) \)

Thus: \( \frac{dt}{dt} = \pm a(t) \frac{d\gamma}{dt} \), i.e., \( \frac{d\gamma}{dt} = \pm \frac{1}{a(t)} \)

Thus: \( d_p = \int_{t_0}^{t} \frac{1}{a(t)} \, dt \) (It’s the comoving distance travelled, and with \( a(t_0) = 1 \),

it’s also the current proper distance to what ultimately we can see.)

For example, for us today: \( d_p \approx 4 \times 10^{10} \) light years. (See some CMB emission)

Recall event horizon:

An observer’s event horizon is the boundary of the past of this observer’s future infinity.

\( \Rightarrow \) If we have a cosmological event horizon, it is the particle horizon that we will have at future infinity.

Q: Do we have a cosmological event horizon?

A: Depends on behavior of \( a(t) \) for \( t \to \infty \):

i.e., does \( d_p = \int_{t_0}^{\infty} \frac{1}{a(t)} \, dt \) converge to a finite comoving distance?
Recall: $a(t) \sim t^{\frac{2}{3(1+w)}}$

$\Rightarrow \quad d_p^\infty = \int_0^\infty \frac{1}{a(t)} \, dt \sim \int_0^\infty t^{\frac{2}{3(1+w)}} \, dt$

$\Rightarrow \exists$ Event horizon iff $w < -\frac{1}{3}$, i.e., if "inflation", i.e., iff $a > 0$!

Notice: $d_p(t) = \int_0^t \frac{1}{a(t')} \, dt' = \tau(t) =$ conformal time!

$\Rightarrow\quad d_p^\infty = d_p(t=\infty)$ is finite $\iff \quad \tau(t=\infty)$ is finite

$\Rightarrow$ If inflation then Penrose diagram truncated above at a $\tau = \tau_{\text{max}}$ line.

Recall:

FL spacetime with $K = 0$, big bang and no late inflation.
Now with inflation:

(as we have today and presumably in the future)

FL spacetime
with \( K = 0 \),

big bang and late inflation:

\[ r = 0 \]

In yellow: The event horizons of observers \( A_1, A_2, A_3, \ldots \) that stay forever causally disconnected.

Line of \( T = T_{\text{max}} \).

\[ \mathcal{I}^- \]

\[ \mathcal{I}^+ \]

\[ \mathcal{I}^- \text{ speciﬁc} \]
Q: Can we analyze the causal structure using a Penrose diagram, i.e., a conformally equivalent diagram whose light rays are at ±45°?  
Q: i.e., is the metric conformally equivalent to Minkowski space?  
Q: Also, can we include the full dynamics of the black hole?  
A: Yes, if we consider only the r,t plane. Why?

**Theorem:** The metric \( g_{\mu\nu} \) of any 2-dimensional Lorentzian manifold or sub-manifold reads in suitable coordinates:

\[
g_{\mu\nu}(x) = \mathcal{E}(x) g_{\mu\nu}
\]

\( \mathcal{E} \) scalar function

Why? In 2D, \( g_{\mu\nu}(x) \) has 3 independent entries and 2 of them can be fixed by choosing the 2 coordinate change functions \( x_1, x_2 \).

\( \Rightarrow \) Every 2D Lorentzian submanifolds of any 3+1 metric has a Penrose diagram.

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**Example:** Collapsing star, forming black hole (non-rotating)

![Penrose diagram](image)

The (light-like) event horizons of all observers who travel to \( i^- \), i.e., who do not fall into the black hole and who do not end up on \( i^+ \), i.e., who do not speed away at the speed of light.

For the transformation, see, e.g., the text by Susskind and Lindsay.
And if the black hole eventually radiates away:

- Singularity (spacelike)
- Similar to Minkowski space again
- $J^+$ "Scri plus"
- $J^-$ "Scri minus"
- Event horizon
- May be fired here? (Scriwell?)
- Unwise traveller