Classification of solutions of GR

Recall:

- The task is to solve the equations of motion of matter, jointly with the Einstein equation:

\[ G_{\mu \nu} = 8\pi G T_{\mu \nu} \]

- In practice, this problem must be simplified, i.e., the number of to-be-determined functions must be reduced.

→ Make symmetry assumptions.

Question: How much can we weaken the symmetry assumptions of Friedmann-Lemaître and still get exact solutions?

Strategy:

- Classify cosmological models \((M, g), T_{\mu \nu}\) by the amount and type of symmetry assumed.

- For each amount and type of symmetry assumed, try to find exact solutions or at least (asymptotic) properties of exact solutions.

Remark: Among the high symmetry models, some come arbitrarily close to F.L. at finite times!

See, e.g., text by Wainwright & Ellis.
Recall: Symmetries & Killing vector fields

- Two spacetimes $(M, g)$, $(\tilde{M}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi: M \rightarrow \tilde{M}$ so that the image of the metric $g$ in $\tilde{M}$ is $\tilde{g}$: $\phi^*g = \tilde{g}$.

- A space-time has a symmetry, if we find such a $\phi$ for $\tilde{M} = M$.

- Example:

  \[ \phi \text{ performs a rotation of } M \text{ about a symmetry axis, to obtain } \tilde{M} = M \text{ with } \phi^*g = \tilde{g}. \]

Note: The set of all symmetries of a manifold $(M, g)$ forms a “group”:

**Definition:** A “group” $G$ is a set, with an operation, say $\circ$, $G \times G \rightarrow G$

and a “neutral element”, say $e$, $e \in G$, such that

- $(a \circ b) \circ c = a \circ (b \circ c)$ $\forall a, b, c \in G$
- $a \circ e = e \circ a = a$ $\forall a \in G$
- $\exists e'$: $a \circ e' = e$ $\forall a \in G$ \text{ “there exists”}

**Example:** The set of rotations in $\mathbb{R}^3$ forms a 3-dimensional Lie group, $SO(3)$. The angles $\alpha, \beta, \gamma$ are coordinates for elements $g \in SO(3)$. 
Remarks: The symmetries of a manifold \((M, g)\) can be discrete, such as reflections.
- But often, the symmetry group of a manifold \((M, g)\) is actually a Lie group.

Note: Each \(h \in G\) yields an isometric diffeomorphism, by assumption.
\[
h : M \to M, \text{ namely } h : p \mapsto h(p) \quad \forall p \in M
\]
- Consider the set \(O_p \subset M\) defined by: \(O_p = \{q \in M | \exists h \in G : h(p) = q\}\).

Definition: The set \(O_p\) is called the Orbit of \(p\) under the action of the group \(G\).

Note: If \(G\) is a Lie group then each orbit \(O_p\) is a proper submanifold of \((M, g)\).

Question: What are the infinitesimal isometric diffeomorphisms? And what type of mathematical structure do the infinitesimal symmetries form?

Recall: The Lie derivative,
\[
\mathcal{L}_\xi Q_{\alpha \cdots k} = Q_{\alpha \cdots k}^{\alpha \cdots k} - Q_{\alpha \cdots k}^{k \cdots \alpha} \xi^k - \cdots - Q_{\alpha \cdots k}^{\alpha \cdots k} \xi^\alpha + Q_{\alpha \cdots k}^{\alpha \cdots k} \xi^\alpha + \cdots + Q_{\alpha \cdots k}^{\alpha \cdots k} \xi^\alpha
\]
yields the rate of change of a tensor \(Q\) along the flow of diffeomorphisms \(\phi\) generated by a vector field \(\xi\).

\(\Rightarrow\) Here, can use \(\mathcal{L}_\xi\) to differentiate along symmetry group orbits.

Thus, if \(\mathcal{L}_\xi g_{\mu \nu} = 0\)
then \(\xi\) generates isometries \(\phi : M \to M, \, g \to \tilde{g} = g\).
But \( \mathbf{g}_{\mu \nu} = 0 = \mathbf{g}_{
u \mu} + \mathbf{g}_{\mu \nu} \), i.e., in itself an infinitesimal symmetry.

\[ \Rightarrow \text{A vector field } \xi \text{ generates a symmetry of spacetime if it is a Killing vector field:} \]
\[ \xi_{\mu} \rightleftharpoons \xi_{\nu} ; \mu = 0 \quad (X) \]

Q: Maximum number, \( d \), of Killing vector fields in \( n \) dimensions?

A: \( d = n(n+1)/2 \) 
To see this, note that there are 2 ways to obey Eq. (X):

\[ \begin{align*}
\text{a) } & \xi_{\mu ; \nu} = 0 \quad \forall \nu, \text{ i.e. } \xi_{\nu} = 0 \\
& \text{can have maximally } n \text{ such independent vectors} \\
\Rightarrow & \quad d = n(n-1)/2
\end{align*} \]

\[ \begin{align*}
\text{b) } & \xi_{\nu} \neq 0, \text{ but then } K_{\nu} := \xi_{\mu ; \nu} \text{ is antisymmetric} \\
& \text{can have at most } n(n-1)/2 \text{ independent such cases.} \\
\Rightarrow & \quad d = n(n+1)/2
\end{align*} \]

From a symmetry Lie group to a "symmetry Lie algebra":

Central idea:

\( \text{Normally the points of a manifold cannot be multiplied!} \)

\( \text{A Lie group is a smooth manifold with extra structure: the multiplication.} \)

\( \text{Notice: Product of group elements close to } 1 \in G \text{ yields a group element close to } 1. \)

\( \text{Consider the tangent space } T_1(G) \text{ to the point } 1 \in G \text{ of the Lie group manifold } G. \)

\( T_1(G) \text{ is a vector space and it has extra structure, inherited from the group's multiplication.} \)

\( \text{Define the Lie algebra of a group } G \text{ to be } T_1(G), \text{ equipped with the inherited "multiplication."} \)

\( \text{The identity element of the group } p = 1 \text{ is also a point of the group's manifold } \)
\( T_1(G) \text{ is the tangent space to this point.} \)

Crucial fact: From knowledge of only the Lie algebra, i.e., only \( T_1(G) \) and its "multiplication," the group \( G \) can be reconstructed!

(though not always uniquely)
Let us collect the properties that the inherited multiplications of all Lie algebras share.

Then, let us define Lie algebras as anything with these properties:

**Definition:**
A Lie algebra is a vector space $A$, with an operation $\{,\}$

\[ \{,\} : A \times A \to A \quad \text{“Lie bracket”} \]

obeying

\[ \{ r, s \} = -\{ s, r \} \quad \forall r, s \in A \quad \text{“Jacobi identity”} \]

and

\[ \{ \{ r, s \}, t \} + \{ \{ s, t \}, r \} + \{ \{ t, r \}, s \} = 0 \]

**Theorem:** Every vector space $A$ with a "multiplication" $\{,\}$ that obeys these axioms is isomorphic to $T_e(G)$ of a Lie group $G$.

**Proposition:** The set of Killing vector fields $\xi^{(i)}$ of $(M,g)$ is a Lie algebra.

**Exercise:** Prove this, i.e., show the following:

Assume $\xi^{(1)}$, $\xi^{(2)}$ are Killing vector fields of $(M,g)$ and $\alpha, \beta \in \mathbb{R}$.

Then:

\[ d\xi^{(1)} + \beta \xi^{(2)} \]

(i.e., they form a linear space)

and

\[ \{ \xi^{(1)}, \xi^{(2)} \} := \xi^{(1)} \xi^{(2)} - \xi^{(2)} \xi^{(1)} \]

are also Killing vector fields, and the $\xi^{(i)}$ obey the Jacobi identity.
Summary of the big picture:

1. The symmetries of any (M,g) form a group: they can be concatenated associatively, and all possess an inverse. Some symmetries are differentiable, parameterized by the flow ⇒ the symmetries form a Lie group.

2. Each Killing vector field is the infinitesimal generator of a flow of isometric diffeomorphisms, i.e., of a symmetry.

3. We see here that the Killing vector fields indeed form a Lie algebra.

4. Recall that every Lie algebra generates a Lie group.

Surfaces of homogeneity and the isotropy subgroup:

- Definition:
  
  Let r be the dimension of the Lie algebra, i.e., also the dimension of the Lie group of symmetries.

- Recall this definition:
  
  Consider the set of points O(p) that a point p can flow to along the Killing vector fields.

  O(p) is called the orbit of p ∈ M under the action of the symmetry group. We denote the dimension of the orbit by s.
Clearly:
The dimension of an orbit cannot be larger than the dimension of the symmetry group, i.e.,

\[ s \leq r \]

but \( s < r \) easily happens:

Example:

- Consider \( M := \mathbb{R}^2 \) and \( p = (0,0) \).

- Then \( r = r_{\text{max}} = \frac{n(m+1)}{2} = 3 \), dim. of sym. group.

\[ \Rightarrow \] The three-dimensional Lie algebra of Killing vector fields is spanned by three Killing vector fields:

Concretely:

\[ K^{(0)} := \frac{\partial}{\partial x}, \quad K^{(2)} := \frac{\partial}{\partial y} \]

\[ K^{(3)} := \gamma \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \]

Group elements generated by them are \( e^{tK^{(0)}} \) and they act as \( e^{tK^{(0)}}(x, y) = (e^{t}x, y) \) by Taylor expansion.

- Orbit of \( p = (0,0) \):

\[ O(p) = \mathbb{R}^2 \] because generators \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) generate flow to every where.

Def: The surface of homogeneity has dimension \( s = 2 \leq r \).

- Notice: Since \( n = 2 \), at any given point \( p \), only at most 2 Killing vectors can be linearly independent at \( p \).
A Role of $K^{(3)}$?

The flow generated by $K^{(3)}$ leaves $p$ fixed and rotates everything around $p$.

A Definition:

We say that those Killing vector fields, which do not generate a homogeneity surface, i.e., which generate a trivial group orbit for a point, are generating the isotropy subgroup (of the full symmetry group generated by all Killing vectors).

A Dimension, $d$, of the isotropy subgroup?

Clearly: $d = r - s$

Classification of cosmological models

The classification is with respect to:

A Dimension of isotropy subgroup $d$:

$0$, $1$, $2$, $3$, $4$, $5$, $6$

e.g. full Lorentz symmetry

A Dimension of homogeneity surfaces $s$:

$0$, $1$, $2$, $3$, $4$

Inhomogeneous

Homogeneous

on 3-dim orbits
A large body of literature exists on most cases of $(d, s)$:
- Many exact solutions are known!
- Many asymptotic behaviors are known!

A comprehensive text:

Wainwright & Ellis, Dyn. systems in cosmology,
Cambridge Univ. Press (1997)

**Examples:**

<table>
<thead>
<tr>
<th>homogeneity</th>
<th>isotropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>d</td>
</tr>
</tbody>
</table>

- 4 3 Einstein's static model
- 4 1 Gödel's model
- 4 0 Osservi-Town models
- 3 3 Friedmann-Lemaître models
- 3 1 Spatially hom & locally one rot. sym.
- 3 0 Bianchi models

**Powerful alternative classification approach:**

John: Classify the possible $T_{ab}$, then use Einstein equation to obtain classification of curvature.

**Proposition:**

For every physical energy momentum tensor $T_{ab}$ there exists a unique timelike vector field $u^a$ so that $T_{ab}$ takes this standard form:

$$T_{ab} = \mu u^a u_b + q_a u_b + q_b u_a + \rho (g_{ab} + u_a u_b) + \Pi_{ab}$$

where $q$ and $\Pi$ are a vector field and a tensor field obeying:

$$q_a u^a = 0, \quad \Pi_{ab} u^b = 0, \quad \Pi_{a}^{a} = 0, \quad \Pi_{ab} = \Pi_{ba}$$
Definition: \( u \) is called the "fundamental 4-velocity field".

Note: E.g., for a perfect fluid this is the fluid velocity:

\[ T_{ab} = \mu u_a u_b + p (g_{ab} + u_a u_b), \quad u_a u^a = -1 \]

Recall: equation of state is

\[ p = (\gamma - 1) \mu \]

\[ \gamma = \begin{cases} 
1 & \text{dust} \\
\frac{4}{3} & \text{radiation} \\
0 & \text{cosmological constant}
\end{cases} \]

Definition:

If \((M, g)\) possesses spatial \( s = 3 \) homogeneity but the fundamental velocity is not orthogonal to the homogeneity surface, then we say that 

this cosmology is "tilted".

Segré classification:

- A systematic classification of \( T_{\mu \nu} \) can be performed, by the analysis of its eigenvalues/eigenvectors. Nontrivial because:

- \( T_{\mu \nu} \) is symmetric.
  
  But, the inner product in the vector space is \( g_{\mu \nu} \Rightarrow T_{\mu \nu} \) is generally not hermitian!

- \( T^\circ \) is in a space with the inner product \( g^\circ_{\mu \nu} = 5 \cdot g_{\mu \nu} \), but \( T^\circ \) is generally not symmetric!

Use Jordan normal form:

\[ \Rightarrow \text{Segré classification yields } 4 \text{ main types of energy momentum tensors } T_{\mu \nu}. \]
Recall strategy:

The classification of possible $T_{\mu\nu}$ should, via the Einstein eqns, yield a classification of possible curvatures.

Indeed: In 3+1 dimensions the Einstein equation also reads:

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

Exercise: Prove this and notice the dimension dependence.

$\Rightarrow$ The 10 degrees of freedom of $T_{\mu\nu}$ (as a symmetric 4x4 matrix) determine the 10 degrees of freedom of $R_{\mu\nu}$.

$\Rightarrow$ The Segré classification of possible $T_{\mu\nu}$ yields, via the Einstein equation also a classification of possible Ricci tensors $R_{\mu\nu}$.

Q: Does this yield also a classification of the possible Riemann tensors $R^{\nu}_{\mu\nu}$?

A: No! The Ricci tensor contains only 10 of the 20 degrees of freedom of the Riemann tensor! (In 3+1 dim)

Prop.: The information in $R^{\nu}_{\mu\nu}$ is shared among the Ricci tensor $R_{\mu\nu}$ and the so-called Weyl tensor $C^{\nu}_{\mu\nu}$.

$\Rightarrow$ It remains to classify the possible Weyl tensors!
The Weyl tensor, $C^{\alpha\beta\gamma\delta}$:

\[ C^{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta} - \frac{1}{2} \left( g^{\alpha \delta} R^{\beta\gamma} + g^{\alpha \gamma} R^{\beta\delta} - g^{\alpha \gamma} R^{\beta\delta} - g^{\alpha \delta} R^{\beta\gamma} \right) + \frac{1}{6} (g^{\alpha \delta} g^{\beta\gamma} - g^{\alpha \gamma} g^{\beta\delta}) R \]

**Notation:** If $R^{\alpha\beta}$ and $C^{\alpha\beta\gamma\delta}$ are given, they determine $R^{\alpha\beta\gamma\delta}$ fully:

\[ R^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta} + \frac{1}{2} \left( g^{\alpha \delta} R^{\beta\gamma} + g^{\alpha \gamma} R^{\beta\delta} - g^{\alpha \gamma} R^{\beta\delta} - g^{\alpha \delta} R^{\beta\gamma} \right) - \frac{1}{6} (g^{\alpha \delta} g^{\beta\gamma} - g^{\alpha \gamma} g^{\beta\delta}) R \]

\[ R^{\alpha\beta\gamma\delta} \text{ is expressed through } C^{\alpha\beta\gamma\delta} \text{ and } R^{\alpha\beta}. \]

\[ \Rightarrow \text{The Weyl tensor } C^{\alpha\beta\gamma\delta} \text{ indeed contains all that information about the curvature } R^{\alpha\beta\gamma\delta} \text{ which is not in } R^{\alpha\beta}. \]

\[ \text{Determines } T_{\mu\nu} \text{ via the Einstein equation.} \]

\[ \Rightarrow C^{\alpha\beta\gamma\delta} \text{ contains all that curvature information which is not determined via the Einstein equation by } T_{\mu\nu}. \]

\[ C^{\alpha\beta\gamma\delta} \text{ describes all that curvature which can exist even where there is no matter! (e.g. gravity waves).} \]

**Proposition**

1. Assume $(M,g)$ is a 3+1 dimensional Lorentzian manifold.
2. Choose any smooth positive scalar function $\phi \text{ on } M$.
3. Define $(\tilde{M},\tilde{g})$ with the new metric obtained through the

\[ g_{\mu
u}(x) \rightarrow \tilde{g}_{\mu
u}(x) := \phi(x) g_{\mu
u}(x) \]

Then: $\tilde{C}^{\alpha\beta\gamma\delta}(x) = C^{\alpha\beta\gamma\delta}(x)$ $\forall x \in \tilde{M}$

(Imagine what would be postulated)
Historical remark

- Consider the equivalence class of spacetimes \((M, g)\) that are conformally equivalent to Minkowski space:
  \[ g_{\mu\nu}(x) = \phi^2(x) g_{\mu\nu} \]
  
- Einstein and Fokker initially considered a theory in which the metric possesses only this conformal degree of freedom \(\phi\) (to play role of Newton's gravitational potential).

Then, \[ S = \frac{1}{2} \mu_0 \left( \sum_{i=1}^{3} V_i^2 dx_i + \sum_{i=1}^{3} \mu_0 V_i^2 dx_i \right) \]
and \[ \frac{\delta S}{\delta \phi} = 0 \]
yield:

\[ R = \Box \phi - 2 \frac{\partial^2 \phi}{\partial t^2} + \frac{\mu_0^2}{c^4} \left( \sum_{i=1}^{3} \mu_0 V_i^2 dx_i \right) \beta_0 + \frac{\mu_0^2}{c^4} \left( \sum_{i=1}^{3} \mu_0^2 V_i^2 dx_i \right) \eta_0 \]

In electromagnetism, \(\beta_0 = 0\), \(\eta_0 = 0\), i.e., EM fields would not grow to be

- Equivalence principle ok.
- Light bending & Mercury perihelion shift wrong.

Recall: via the Einstein equation the Segre classification implies a classification of properties of the Ricci tensor \(R_{\mu\nu}\).

It remains to classify the Weyl tensor:

Petrov classification:

This is a classification of the Weyl tensor \(C^{\mu\nu}_{\rho\sigma}\), which possesses the 10 remaining degrees of freedom of \(R_{\mu\nu}\):

- \(\mathcal{C}^{\mu\nu}_{\rho\sigma}\) just like the Riemann tensor, is antisymmetric in \(\mu, \nu\), \(\rho, \sigma\), and symmetric in \(\rho, \sigma\).
Thus $C_{\gamma\gamma}^{\gamma\gamma}$ can locally be viewed as a symmetric map from the antisymmetric part $A_p(\mathbb{R}^2)$ of $T_p(\mathbb{R}^2)$ (so called bi-vectors) into itself:

$$C : A_p(\mathbb{R}^2) \rightarrow A_p(\mathbb{R}^2)$$

But the inner product in $A_p(\mathbb{R}^2)$ is not positive definite!

$$\Rightarrow C$$ is generally not hermitian. Therefore, use Jordan normal form again:

Result: 6 main Petrov classes for Weyl curvature:

- According to eigenvalue/eigenvector decomposition:
  - **Type O**: Weyl curvature vanishes
  - **Type D**: "Static" Weyl curvature, e.g. in vicinity of a star.
  - **Type N**: Transverse gravitational waves, the type LIGO aims to detect. Like light, their strength decays $\sim \frac{1}{r^2}$ from the source.
  - **Type I**: Longitudinal gravitational waves. These waves cause a shear effect. However, they decay fast: $\sim \frac{1}{r^2}$
    - Why? Gravitational waves, when small enough, travel with speed of light. Like light, they then cannot oscillate longitudinally.

- **Types II, III**: Mixtures of the above.
Potential problem: (with symmetry assumptions):

- The so-obtained highly symmetric solutions, e.g. Friedman-Lemaître, may possess properties that are peculiar to high symmetry.

- E.g.: When a Friedman-Lemaître solution, or a Schwarzschild solution exhibits a singularity: Is it due to symmetry, or realistic?

- Singularity theorems (see later) confirm the robustness under certain conditions (such as strong energy condition).

→ More confidence in significance of the properties of highly symmetric solutions.