A singularity theorem:

Assume that:

- \((M,g)\) is a globally hyperbolic spacetime
- \(\text{The energy-momentum tensor of matter obeys the Strong energy condition:}\)
  \[
  (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \varepsilon^{\mu\nu} \geq 0 \quad \forall \text{ timelike } \varepsilon.
  \]
- \(\text{There exists a } C^2 \text{ spacelike Cauchy surface } \Sigma,\)
- \(\text{on which the trace of the extrinsic curvature, } K, \text{ is bounded from above by a negative constant } C:}\)
  \[
  K(p) \leq C < 0 \quad \text{for all } p \in \Sigma.
  \]

Then:

- No past-directed timelike curve from a spacelike hypersurface \(\Sigma\) can have eigentime, i.e., proper length, larger than \(\frac{3}{C}\).
- J.e.: All past-directed timelike geodesics are incomplete.

\[ \Rightarrow \text{There is a cosmological singularity in the finite past!} \]
Extrinsic curvature?

- The **extrinsic curvature** of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

  **Intuitively**: It is the rate of the expansion of spacetime, more precisely its negative, the rate of contraction.

Thus: Assuming \( K(p) \leq C < 0 \) meant that spacetime has a finite minimum expansion rate everywhere on \( \Sigma \).

\( \Rightarrow \) We'll define expansion below in detail.

---

The strong energy condition?

**Recall**: The "weak energy condition":

\[
T_{\mu\nu} v^{\mu} v^{\nu} \geq 0 \quad \text{for all timelike } v : g(v,v) < 0
\]

**Meaning?** For an observer with unit tangent \( v \) the local energy density is: \( T_{\mu\nu} v^{\mu} v^{\nu} > 0 \)

- The "dominant energy condition":

\[
T_{\mu\nu} v^{\mu} v^{\nu} \geq 0 \quad \text{and} \quad K_{\mu} k^{\mu} \leq 0
\]

**Meaning?** The local energy-momentum flow vector \( K \) may not be conserved but has to be non-space-like: flow should be into the future - need for causality.
The "strong energy condition"

Matter is said to obey the strong energy condition iff:

\[(T_{\mu\nu} - \frac{1}{2} T^{\rho}_{\rho} g_{\mu\nu}) g^{\mu\nu} \geq 0 \quad \text{forall timelike } \xi.\]

As we will discuss below

Intuition?: Exclude matter that causes accelerated expansion.

Plausible?: Yes, obeyed by known matter. (but not by dark energy)

Relationship?: Independent of weak and dominant energy conditions.

Concretely: For known matter, \(T_{\mu\nu}\) is diagonalizable to obtain:

\[T_{\mu\nu} = \begin{pmatrix} \rho & p & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}\]

The energy conditions then read:

- Weak: \(\rho > 0\) and \(\rho + p > 0\) for \(i \in \{1,2,3\}\)

- Dominant: \(\rho \geq 1 p; 1\) for \(i \in \{1,2,3\}\)

Exercise: Show this \[\text{Strong: } \rho + \sum_{i=1}^{3} p_{i} > 0 \quad \text{and} \quad \rho + p > 0 \quad \text{for } i \in \{1,2,3\}\]

Recall: A cosmological constant \(\Lambda\) can be viewed as a contribution to \(T_{\mu\nu}\).

Indeed, below is no big energy singularity \(p_{i} < 0 \forall i\), i.e., in the Einstein equations, replace \(\phi_{c} = \phi_{c} + \Lambda\).

Exercise: Show that the strong energy condition is violated in cosmology iff \(w < -\frac{1}{3}\), i.e., iff the expansion is accelerating: \(a(t) > 0\).
Essence of point c):

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, \( \Theta \), in finite proper time:

\[ \text{The "expansion", } \Theta: \]

- Consider a "congruence of timelike geodesics" through \( \Sigma \), i.e., a smooth family of timelike geodesics, exactly one through each \( p \in \Sigma \). If parametrized by proper time, their tangent vector field \( \xi \), namely

\[ \xi := \frac{d}{d \tau} \text{ proper time} \]

will obey: \( g(\xi, \xi) = -1 \) \( \forall p \).

- Consider now a one-parameter subfamily of these geodesics:

\[ \gamma^i(\tau, s) \]

parametric family of family of neighboring geodesics.

The "connecting vector field"

Then, we define the deviation vector:

\[ \eta := \frac{d}{ds} \]

- a line of constant \( \xi \) value

- a geodesic, i.e., a line of constant \( s \) value
How does $\eta$ change along a geodesic? $\tau, s$ are Riemann normal coordinates for a geodesic traveller.

$$\Rightarrow \frac{D}{ds} \frac{D}{\tau} = \frac{D}{\tau} \frac{D}{ds}, \text{ i.e., } \{\phi, \eta\} = 0$$

Since the torsion vanishes: $0 = \mathcal{T}(\xi, \eta) = \xi^a \eta_b - \eta^a \xi_b - [\xi, \eta]$

$$\Rightarrow \xi^a \eta_b = \eta^a \xi_b$$

$$\Rightarrow \xi^a \xi_b = \eta^a \eta_b$$

$$\Rightarrow \xi^a \eta_b = \xi^a \eta_b$$

$$\Rightarrow \xi^a \xi_b = \eta^a \eta_b$$

$$\Rightarrow \xi^a \xi_b = \eta^a \eta_b$$

$$\Rightarrow B^{\lambda}_{\mu} \xi^\mu = \eta^\mu$$

Along the geodesic's direction, $\xi$, the deviation vector $\eta^\mu$ changes its direction and length by $B^{\lambda}_{\mu} \eta^\mu$.

The tensor $B^{\lambda}_{\mu}$ can be decomposed covariantly and uniquely into:

$$B^{\lambda}_{\mu} = \omega^{\lambda}_{\mu} + \epsilon^{\lambda}_{\mu}$$

(All 3 terms are tensors because the split is covariant)

We have: $\omega^{\lambda}_{\mu} = \frac{1}{2} \left( B^{\lambda}_{\mu} - B^{\mu}_{\lambda} \right)$, clearly.

But $\epsilon^{\lambda}_{\mu}$, $\epsilon^{\mu}_{\lambda} = 0$.

In preparation: define the projector $h_{\mu \nu}$ onto $(R S)^\perp$, i.e. onto the spatial components:

$$h_{\mu \nu} := g_{\mu \nu} + \eta_{\mu} \eta_{\nu}$$

Check: is $h_{\mu \nu} \xi^{\nu}$ really always 1 to 1 to $\xi$?

Indeed: $\xi^{\nu} h_{\mu \nu} \xi^{\mu} = (\xi, \xi) + (\xi, \xi)(\xi, \xi) = 0$
Define: The "expansion", \( \Theta \), is defined as the magnitude of the spatial part of \( B \):
\[
\Theta = B^{\mu\nu} h_{\mu\nu}
\]

Claim: \( \text{Tr}(B) = \Theta \)

Indeed:
\[
\Theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \delta^{\mu}_{\nu} \delta^{\nu}_{\mu} B_{\mu\nu}
\]
\[
= \text{Tr}(B) + \delta^{\mu}_{\nu} g_{\mu\nu} (\text{zero because } g_{\mu\nu} = 0)
\]
(verify)

Therefore:
\[
\alpha_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu} - \frac{1}{3} \Theta h_{\mu\nu})
\]
(verify)

\( \Rightarrow \) the part of \( B_{\mu\nu} \) which is symmetric and traceless.

and:
\[
\beta_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} \quad \text{is the "first term".}
\]

- Interpretation:

  a) \( \omega_{\mu\nu} \) is antisymmetric: \( \omega_{\mu\nu} = -\omega_{\nu\mu} \)

  \( \Rightarrow \) it generates Lorentz transformation for \( \gamma \).

  but all \( \gamma \) are \( \perp \) to the time direction

  \( \Rightarrow \) \( \omega_{\mu\nu} \) generates spatial rotations of neighboring geodesics around another. So, \( \omega_{\mu\nu} \) is called

  \( \omega = "\text{Trends tensor"} \)

  One can prove: (non-trivial)

  If one chooses the congruence of geodesics \( \perp \Sigma \) then \( \omega_{\mu\nu} = 0 \).
b) $\delta_{\mu\nu}$ is symmetric, $\delta_{\mu\nu} = \delta_{\nu\mu}$. (i.e. hermitian)

Consider "diagonalized", by suitable choice of $cd$ basis.

$\Rightarrow$ $\delta_{\mu\nu}$ changes the relative lengths of the basis's vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

$\Rightarrow$ The volume spanned by basis's vectors stays the same under the action of $\delta$.

$\Rightarrow$ **Definition**: $\dot{\gamma}_\nu = \text{"Shear tensor"}$

\[ \dot{\gamma}_\nu = \frac{1}{3} \Theta h_{\mu\nu} = \text{"Expansion tensor"} \]

(Recall: is projector on spatial part.)

\[ \text{Is indeed generating the spatial expansion or contraction of nearby geodesics!} \]

\[ \text{Evolution of $\Theta$ along a geodesic?} \]

\[ c) \text{While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, $\Theta$, i.e., more precisely} \]

\[ \Theta_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} = \text{"Expansion tensor"} \]

\[ \text{is indeed generating the spatial expansion or contraction of nearby geodesics!} \]
Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics must achieve one of two things: a bounce off a singularity or become a singularity in finite proper time. The latter will mean a big bang singularity:

- Important notion also by existence of zero energy stars.

**The "expansion", θ:**

- Consider a "convergence of timelike geodesics" (e.g., freely falling dust) through Σ, i.e., a smooth family of timelike geodesics, exactly one through each point p ∈ Σ: (Σ is a Cauchy surface)

**We consider a one-parameter sub-family of these geodesics:**

- \( y(t, s) \)
  - \( t \) parametric family of family of neighboring geodesics.
  - \( s \) constant at a line of constant \( t \) value

- \( s \) a geodesic, i.e., a line of constant \( s \) value

**Then, we define the deviation vector to a neighboring geodesic:**

- \( \eta := \frac{d}{ds} \)

**The singularity theorem claims that this happened in the past:**

- Big bang singularity
How does $\gamma$ change along a past-directed timelike geodesic with tangent $\xi$?

We showed:

$\xi^\nu \xi^\mu = g^{\nu \mu} B_{\nu \mu}$ where $B_{\nu \mu} := \xi^\nu \xi^\mu$

⇒ Along the geodesic, $\xi$, the deviation vector $\eta^\nu$ changes its direction and length by $B_{\nu \mu} \eta^\mu$.

 tensor $B_{\nu \mu}$ can be decomposed covariantly

and uniquely:

$$B_{\nu \mu} = \omega_{\nu \mu} + \Gamma_{\nu \mu} + \psi_{\nu \mu}$$

Symmetric and trace $= 0$

up

antisymmetric

and symmetric

up

up

Explictly:

Volume preserving:

$$\omega_{\nu \mu} = \frac{1}{2} (B_{\nu \mu} - B_{\mu \nu})$$

Twist: $\nabla \omega_{\nu \mu} = 0$  $
abla \psi_{\nu \mu} = 0$

$$\psi_{\nu \mu} = \frac{1}{2} (B_{\nu \mu} + B_{\mu \nu}) = \frac{1}{2} \Theta h_{\nu \mu}$$

Shear: $\nabla \Gamma_{\nu \mu} = 0$

Volume changing:

$$\psi_{\nu \mu} = \frac{1}{3} \Theta h_{\nu \mu}$$

Expansion: $\nabla \Theta = 0$

Here, we defined: $\Theta := B^{\nu \mu} g_{\nu \mu}$ and $h_{\nu \mu} := g_{\nu \mu} + \xi_{\nu} \xi_{\mu}$

i.e., the Expansion, $\Theta$, in the trace of $B$, which we showed is also equal to the magnitude of the spatial part of $B$: $\Theta = B^{\nu \mu} h_{\nu \mu}$.

Key question: What is the dynamics of $\Theta$?
The Raychaudhuri equation

For the derivation, we will use:

A) Definition of $B$ is: $B_{\mu \nu} := \xi_{\mu} \xi_{\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\xi^a_{\ jbc} - \xi^a_{\ jcb} = R^a_{\ bcd} \xi^d$$

C) $\xi$ is tangent to a geodesic, i.e., it obeys: $\frac{D \xi}{D\lambda} = 0$

i.e.: $0 = \nabla_{\xi} e_a \xi^b e_b = \xi^a \nabla_{e_a} \xi^b e_b = \xi^a \xi^b \nabla_{e_a} e_b$

True for all $e_a$, thus: $\xi^a \xi^b \xi_{\ ja} = 0$

Now calculate the rate of change of $B$ along the geodesic:

$$\frac{\partial}{\partial \lambda} B_{\alpha \beta \gamma} = \nabla_{\xi} B_{\alpha \beta \gamma}$$

$$= \xi^c \nabla_{\xi} B_{\alpha \beta \gamma c} + \xi^c \nabla_{\xi} R_{\alpha \beta \gamma \delta} \xi^\delta$$

Using:

$$\frac{\partial}{\partial \lambda} \left( \xi^c \nabla_{\xi} \xi_{\alpha j c} \right) = - \xi^c \xi_{\ jb} \nabla_{\xi} \xi_{\alpha j c} + R_{\alpha \beta \gamma \delta} \xi^c \xi^\delta$$

C) $\frac{\partial}{\partial \lambda} \left( - \xi^c \xi_{\ jb} \nabla_{\xi} \xi_{\alpha j c} + R_{\alpha \beta \gamma \delta} \xi^c \xi^\delta \right)$

(4) $\frac{\partial}{\partial \lambda} B_{\alpha \beta \gamma} = -B^c_{\ jb} B_{\alpha \beta c} + R_{\alpha \beta \gamma \delta} \xi^c \xi^\delta$
In summary, we derived:

\[ \xi^c B_{abc} = -B^c_b B_{ac} + \text{Rade} \xi^c \xi^d \]  

\( (*) \)

The trace of \( (*) \) will be the Raychaudhuri equation.

But first, we recall:

\[ \xi = \frac{\dot{\Theta}}{\Theta} \]

\[ T B = B_{\nu \rho} g^{\nu \rho} = \Theta \]

\( \Rightarrow \text{Trace(LHS) of (\( \star \)) reads } \frac{d}{d\tau} \Theta ! \)

Now on the RHS of \( (*) \) use the decomposition

\[ B_{\mu \nu} = \omega_{\mu \nu} + \sigma_{\mu \nu} + \frac{1}{2} \Theta h_{\mu \nu} \text{ to express } B^c_b B_{ac}; \]

\[ B^c_b B_{ac} = \omega^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{2} \Theta h_{ac}) \]

\[ + \sigma^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{2} \Theta h_{ac}) \]

\[ + \frac{1}{3} \Theta h^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{2} \Theta h_{ac}) \]

When taking the trace, \[ g^{ab} B^c_b B_{ac}, \text{ only the diagonal terms survive:} \]

\[ \text{Tr} (BB) = g^{ab} B^c_b B_{ac} = \omega_{ab} \omega_{ab} + \sigma_{ab} \sigma_{ab} + \frac{1}{6} \Theta^2 h_{ab} h_{ab} \]

The Raychaudhuri equation is then the trace of Eq.\( (*) \):

\[ \frac{d\Theta}{d\tau} = -\frac{1}{3} \Theta^2 - \sigma_{ab} \sigma_{ab} - \omega_{ab} \omega_{ab} - \text{Rade} \xi^c \xi^d \]

\( \Theta \) is always positive. There may be \( 0 \) or \( \pm \) depending on the sign of \( \Theta \).
Dynamics

a) Assume that

\[ R_{\mu \nu} \xi^\mu \xi^\nu \geq 0 \] for all timelike \( \xi \)

i.e., using the Einstein equation

\[ R_{\mu \nu} = 8 \pi G \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right) \]

we are assuming that

\[ T_{\mu \nu} \xi^\mu \xi^\nu - \frac{1}{2} \xi^\mu \xi^\nu T \geq 0 \]

where \( \xi^\mu \xi^\nu < 0 \)

i.e. the **Strong Energy Condition**.

Thus, assuming the strong energy condition:

\[ \frac{d \Theta}{d \tau} + \frac{1}{3} \Theta^2 \leq 0 \]

i.e.,

\[ -\frac{1}{\Theta^2} \frac{d \Theta}{d \tau} - \frac{1}{3} \geq 0 \]

i.e.,

\[ \frac{d}{d \tau} \Theta^{-1} \geq \frac{1}{3} \]

(\( \Theta \))

consider the cases when the geodesics are initially all either

a) diverging, i.e., \( \Theta(\tau_0) > 0 \) (expanding universe) or

b) converging, i.e., \( \Theta(\tau_0) < 0 \) (contracting universe)

(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere).)
We see that $\Theta''(\tau) = 0$ at a finite time $\tau_{BB}$ (Big Bang).

We see that $\Theta'(\tau)$ will hit $\Theta'(\tau) = 0$ at a finite time $\tau_{BC}$ (Big Crunch).

**Conclusion:**

Eq. (4) implies that $\Theta(\tau)$ must go through 0; i.e.:  

a.) for sufficiently early $\tau$, have $\Theta \to \infty$, i.e., Big Bang

b.) for sufficiently late $\tau$, have $\Theta \to -\infty$, i.e., Big Crunch

**Note:** This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.