Note Title

## More on singularity theorems

- Assume a set of symmetries of matter and spacetime has been chosen.
- Assume an exact solution or at least its asymptotic properties at early times have been found.
- Assume, we choose a timelike congruence e.g. of geodesics.

$\Rightarrow$ We can now explicitly calculate the twist, shear and expansion along the congruence:

## The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

- Now we have different expansions in different directions, nonlinearly influencing another.
- Recall:

  The expansion in one direction can be enhanced by shear, as long as shear shrinks other directions.
- Definition:

  We define a rate of expansion tensor that includes shear:

$$\Theta_{\mu\nu} := \underbrace{\sigma_{\mu\nu}}_{\text{shear}} + \frac{1}{3}\,\underbrace{\theta}_{\text{expansion scalar.}}\,\underbrace{h_{\mu\nu}}_{\text{projector } \perp \text{ to the timelike } u\text{-field}}$$

symmetric part of $B_{\mu\nu}$

- $\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & & & 0 \\ & \theta_1 & & \\ & & \theta_2 & \\ 0 & & & \theta_3 \end{pmatrix}$$

3 space-like directions.

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3$$

why $\frac{1}{3}$? Recall that $\underbrace{Tr(h_{\mu\nu}) = 3}$ $\Rightarrow$ is not quite projector

- **Definition:**

$H_i := \frac{1}{3}\theta_i$ Local Hubble expansion function in direction $e_i$.

$H := \frac{1}{3}\theta$ Overall local Hubble expansion function.

- **Definition:**

We use $H_i$, $H$ to define local directional and general scale factors $l_i$, $l$:

The $l_i$, $l$ are defined as the solutions to:

$$\frac{\dot{l}_i}{l_i} = H_i$$

$$\frac{\dot{l}}{l} = H$$

Here, the time derivative is defined as:

$$\dot{l} = u(l) = u^\mu \frac{\partial}{\partial x^\mu} l$$

recall: $u$ is timelike.

- What behavior can occur in the far past?

Full set of cases not yet known.

But:

Explicit examples are known where e.g:

- All $l_i \to 0$ as in FL cosmologies
- $l_1, l_2 \to 0$, $l_3 \to \infty$ "cigar singularity"
- $l_1, l_2 \to 0$, $l_3 \to \text{const}$ "barrel singularity"
- $l_1, l_2 \to \text{const}$, $l_3 \to 0$ "pancake singularity"

- **Note:** For homogeneous, isotropic FL models, $H$ is the regular Hubble parameter and $l$ is its scale factor.

# Singularity theorems for black holes

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

Mass of black hole

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Singularity: $r = 0$

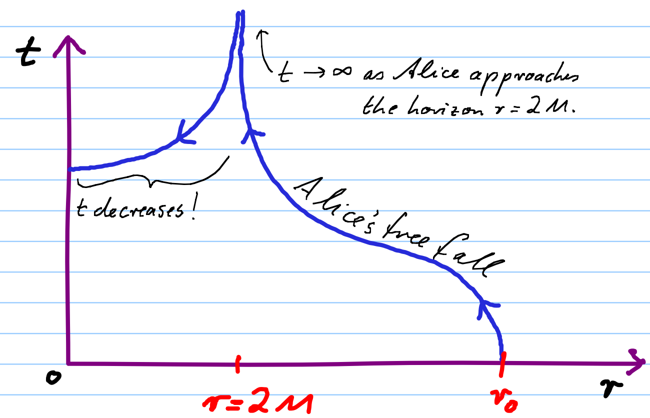
Horizon: $r = 2M$

Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in other coordinate systems. The curvature is smooth across $r = 2M$.

- Due to the sign changes across $r = 2M$, for $r < 2M$ $dt$ is spacelike and $dr$ is timelike for $r < 2M$.
- Consider, for example, a traveler, Alice, who is freely falling from $r = r_0$ to $r = 0$:




$$r(\alpha) = \frac{r_0}{2}\left(1 + \cos(\alpha)\right)$$

$$t(\alpha) = \left(\frac{r_0}{2} + 2M\right) w\alpha + \frac{r_0}{2} w \sin(\alpha) + 2M \log\left|\frac{w + \tan(\alpha/2)}{w - \tan(\alpha/2)}\right|$$

$$\tau(\alpha) = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}\left(\alpha + \sin(\alpha)\right)$$

Here: $0 < \alpha < \pi$ and $w := \left(\frac{r_0}{2M} - 1\right)^{1/2}$

$\Longrightarrow$ need better choices of coordinate systems!

## Simplification:

For now, we drop the $\varphi$ and $\theta$ coordinates.

## First design of a new cds $(T, R)$ - Alice's choice (for $r_0 = 2M$):

- Require $g_{\mu\nu}(T,R)$ to be regular across $r = 2M$.
- Require $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$ at $r = 2M$. If there's really no singularity at $r = 2M$ this must be possible.
- Extend this cds so that $g_{\mu\nu}(T,R) = f(T,R)\,\eta_{\mu\nu}$

⇒ Alice's choice are the Kruskal-Szekeres coordinates $(T,R)$:

$$T(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\sinh\left(\frac{t}{4M}\right)\theta(r-2M)+\cosh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

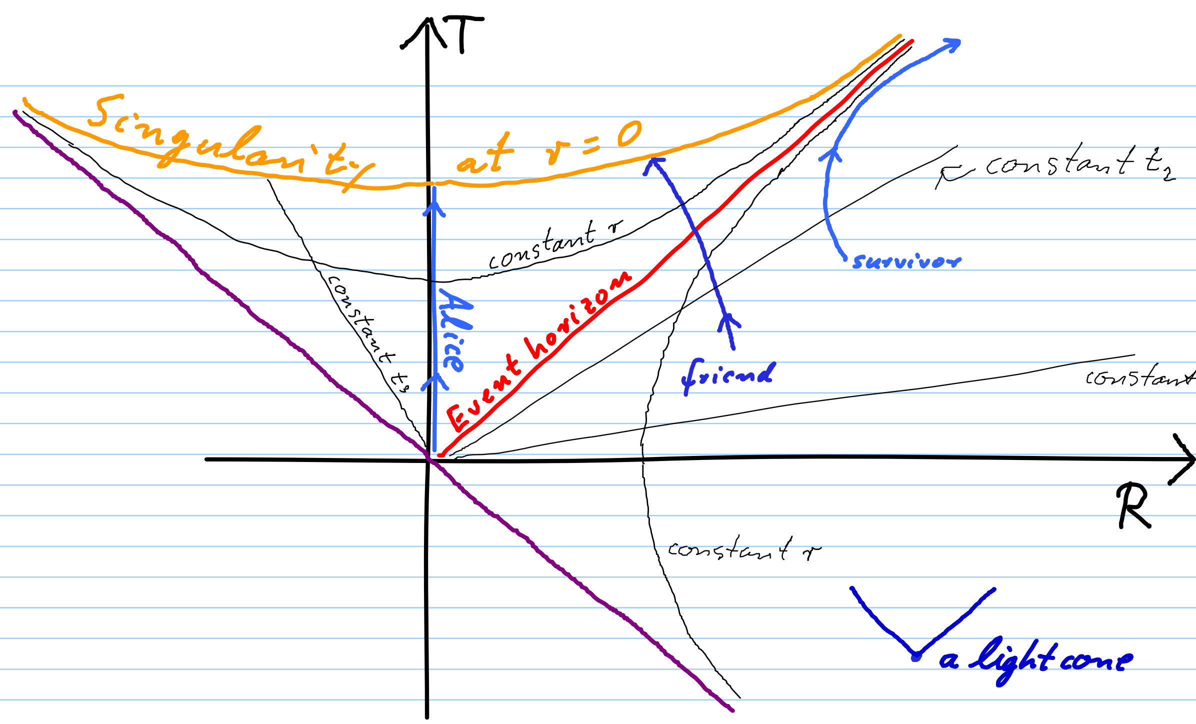
$$R(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\cosh\left(\frac{t}{4M}\right)\theta(r-2M)+\sinh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \underbrace{\frac{2M}{r(T,R)}\, e^{1-\frac{r(T,R)}{2M}}}_{\text{Conformal prefactor}=1 \text{ as } r=2M}\underbrace{\left(dT^2 - dR^2\right)}_{=\eta_{\mu\nu}}$$

Obeys all conditions!




- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

## Alice's light cone coordinates:

$$u := T - R, \quad v := T + R$$

Metric: $ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1-\frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor} \\ \text{(which is 1 at horizon)}}} \underbrace{du\, dv}_{\substack{\text{lightcone} \\ \text{Minkowski}}}$

$\Rightarrow$ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\, d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T \phi(T,R)\right)^2 - \left(\partial_R \phi(T,R)\right)^2 dT\, dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u \phi(u,v)\right)\left(\partial_v \phi(u,v)\right) dv\, du$$

$\leftarrow$ b/c region $T > -R$ means $T+R>0$, i.e. $v>0$.

$\Rightarrow$ Eqn of motion: $\partial_u \partial_v \phi(u,v) = 0$

## Bob's choice of coordinate system

Bob is far from the black hole.

He wants a cds in which:

- $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ as $r \to \infty$.
- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- photons travel at $45°$
- equations of motion of matter fields will be simple (useful in QFT!)

$\leadsto$ Bob's choice is the Tortoise coordinate system.

## Tortoise cds $(t^*, r^*)$:

- In terms of the Schwarzschild cds:

$$t^* := t$$

must require $r > 2M$!

$$r^* := r - 2M + 2M \log\left(\frac{r}{2M} - 1\right)$$

$\Rightarrow$ Important: This is in principle invertible, to obtain

$$r = r(r^*)$$

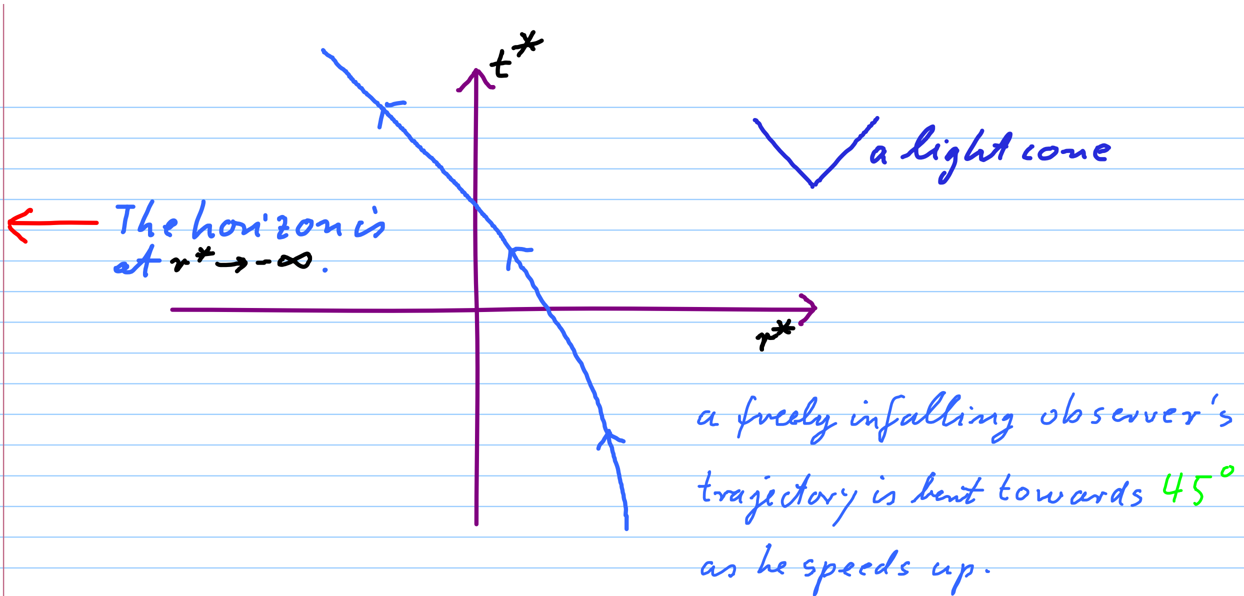
but only for $r > 2M$!

$\Rightarrow$ The tortoise cds only cover the BH's outside!

**Metric:**

$$ds^2 = \underbrace{\left(1 - \frac{2M}{r(r^*)}\right)}\left(dt^{*2} - dr^{*2}\right)$$

Conformal factor $\to 1$ as $r \to \infty$, as planned but $\to 0$ at horizon.




**Bob's light cone coordinates:** $\bar{u} := t^* - r^*, \quad \bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(\bar{u},\bar{v})}\right)}_{\to 1 \text{ as } r\to\infty \text{ and } \to 0 \text{ as } r \to 2M} d\bar{u}\, d\bar{v}$$

## ⇒ The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x$$ becomes:

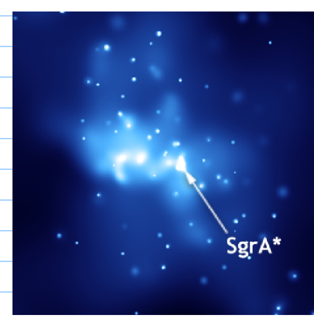
$$= \frac{1}{2}\int_{\mathbb{R}^2}\left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2\,dt^*\,dr^*$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\partial_{\bar{u}}\phi(\bar{u},\bar{v})\right)\left(\partial_{\bar{v}}\phi(\bar{u},\bar{v})\right)d\bar{v}\,d\bar{u}$$

⇒ Eqn of motion: $\partial_{\bar{u}}\partial_{\bar{v}}\phi(\bar{u},\bar{v}) = 0$

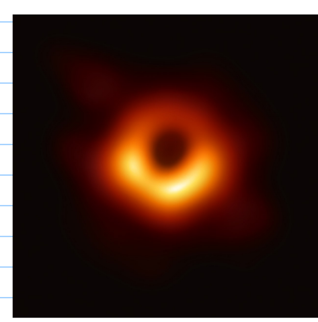
## Do real black holes possess a singularity?

### Sagittarius $A^*$




- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 light years away at centre of Milky Way.

→ Observation of M87 by the Event Horizon Telescope (in mm band) with enough resolution to see the event horizon:



## How to model properties of real black holes roughly?

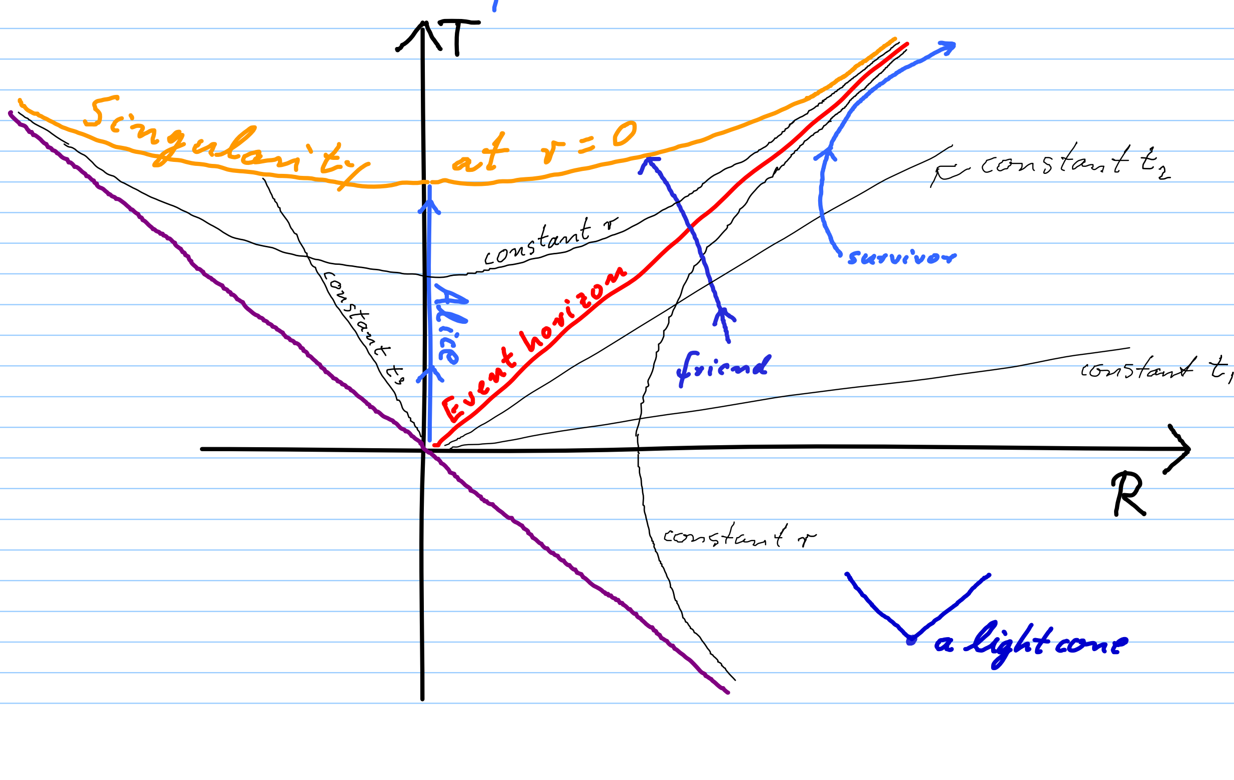
Singularity theorems suitable for black holes involve the concept and assumption of a trapped surface:

**Def:**

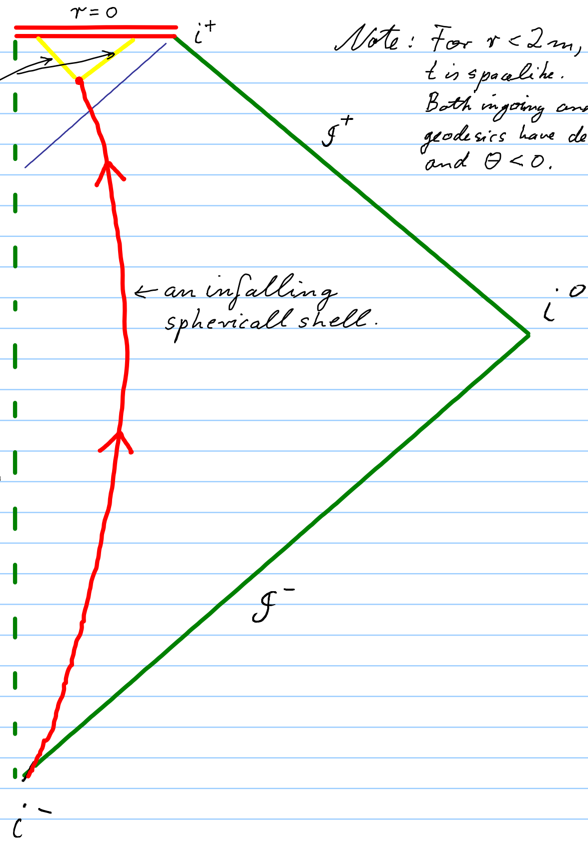
- Let $\Sigma$ be a spacelike hypersurface. (Note: $\Sigma$ is 3-dimensional)
- Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of $\Sigma$. Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to $T$.
- If all these geodesics possess negative expansion, $\theta < 0$, then $T$ is called a trapped surface.

**Examples of trapped surfaces:**

All spheres $r = \text{const.}$ inside a Schwarzschild black hole.



Generally:




Note: For $r<2m$, $r$ is timelike and $t$ is spacelike.
Both ingoing and outgoing null geodesics have decreasing $r$, and $\theta<0$.

The in- and outgoing null geodesics both have negative expansion. Can't see it here b/c the neighboring geodesics are neighbors in the suppressed angular directions.

**Def:** Let $\Sigma$ be a spacelike hypersurface.

Then, the (3-dim. spacelike) union, $\mathfrak{T}$, of all trapped surfaces $T\subset\Sigma$ is called the **trapped region** of $\Sigma$.

**Def:** The boundary $\partial\mathfrak{T}\subset\Sigma$ is called the **apparent horizon** of the spacelike hypersurface $\Sigma$.

Note: $\partial\mathfrak{T}$ is 2-dimensional and spacelike.

**Def:** If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha,\ \alpha\in I\subset\mathbb{R}$$

each with its apparent horizon, $\mathfrak{T}_\alpha$, then their union

$$\mathcal{A}:=\bigcup_\alpha\mathfrak{T}_\alpha$$

is called the **Trapping horizon** of the spacetime.

Remarks:

- To check for the existence of an event horizon $\mathcal{J}^-$ (worldline to $i^+$) in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- The notion of apparent horizons is dependent on the the choice of foliation of spacetime into spacelike hypersurfaces.

- For static Schwarzschild black holes the event and apparent horizons coincide.

- Singularity theorems for black holes make assumptions of apparent horizons.

Comment:

Hawking radiation is usually thought to emanate from the apparent horizon.