Integration

Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

A: Antisymmetry → special transformation property under chart changes:

\[ \sim \det (\text{Jacobian}) \]

⇒ suitable for integration:

- 2-forms have natural integrals in 2-dimensional manifolds
- 3-forms have natural integrals in 3-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

Thus: Before defining integration on manifolds, must study notion of "Orientation" of the manifold.

Namely: Consider e.g., 1-dim manifold: \( \mathbb{R}^1 \)

- Could have charts of the type

- or charts of the type

But, since \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \), one needs to decide!

because \( \frac{dt}{dx} = -1 \) (which is \( \det (\text{Jacobian}) \)
For n-dim mfslds, may need several charts.

**Definitions:**

1. A complete collection of charts, i.e., an **Atlas**, \( \mathcal{A} \), is called **oriented** if among all overlapping charts with coordinates say \( x, \tilde{x} \), the Jacobian determinants are positive:

\[
\det \left( \frac{\partial \tilde{x}}{\partial x} \right) > 0
\]

2. A mfsld \( M \) is called **orientable** if it possesses an oriented atlas.

**Example:** Möbius strips

are not orientable.

- A mfsld, \( M \), together with a choice of oriented atlas, \( \mathcal{A} \), is called an **oriented manifold**.

- Then, an arbitrary chart is called positive (or negative) if its Jacobian determinant with charts of the atlas \( \mathcal{A} \) is positive (or negative).
Definition:
An $n$-form $\Omega \in \Lambda^n(M)$ is called a volume form if it nowhere vanishes. We will later find a preferred volume form for space-time (using the metric).

Proposition:
$M$ possesses a volume form $\Omega$ $\iff$ $M$ is orientable.

Integration:
- Recall change of cds in integration in $\mathbb{R}^n$:

  For $(x', \ldots, x^n) \rightarrow (\tilde{x}', \ldots, \tilde{x}^n)$:

  $\int_{\mathbb{R}^n} g(x', \ldots, x^n) \, dx' \ldots dx^n = \int_{\mathbb{R}^n} g(x(\tilde{x})) \det \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) d\tilde{x}' \ldots d\tilde{x}^n$

  - Jacobian determinant is negative if coordinate system's change hands.

- Now for a general $n$-dimensional differentiable manifold $M$,
  consider an $n$-form $\omega$ in a chart:

  $\omega = \omega(x) \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$
Then what is \( w \) in an overlapping, second chart?

\[
w = f(x(y)) \frac{\partial y}{\partial x^1} \frac{\partial y}{\partial x^2} \cdots \frac{\partial y}{\partial x^n} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n
\]
totally antisymmetric!

- terms are nonzero only if contain each number 1,...,\( n \)
  exactly once, e.g. \( dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \).
- Reorder those terms - they are all
  \( dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \)
up to a possible factor \(-1\) because \( dx^i \wedge dx^i = -dx^i \wedge dx^i \).

\[
\Rightarrow \quad w = f(x(y)) \det\left( \frac{\partial y}{\partial x^i} \right) \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n
\]

Compare with equation (\( \star \)) above \( \Rightarrow \)

The following definition of the integral of \( n \)-forms in an \( n \)-dim. diffeable mfd \( M \) is chart-independent, i.e., is well-defined:

**Definition:**

Assume \( M \) is an oriented \( n \)-dim mfd

and \( \omega \in \Lambda^m(M) \) reads in a chart \( \omega \):

\[
\omega = f(y) \, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m.
\]

Then, if one chart suffices:

\[
\int_M \omega := \int_{\mathbb{R}^m} f(x) \, dx^1 \, dx^2 \cdots dx^m
\]

Else: Piece right hand side together from several charts.

**Note:** how to piece together does not matter as long as charts are from the atlas that \( M \) is equipped with. That’s why orientation is important.
Definition: The boundary operator, \( \partial \)

Assume \( \mathcal{G} \subset M \) is a region (i.e., an \( n \)-dim. open and connected subset) of the \( n \)-dim. manifold \( M \).

We denote the \( (n-1) \)-dim. boundary manifold of \( \mathcal{G} \) by \( \partial \mathcal{G} \):

\[ \partial \mathcal{G} := \text{boundary}(\mathcal{G}) \]

We say that \( \partial \mathcal{G} \) is smooth if locally there exist charts so that:

Proposition: If \( M \) is orientable, then so is \( \mathcal{G} \). Also, the orientation of \( \mathcal{G} \) induces an orientation of \( \partial \mathcal{G} \).

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If \( \partial \mathcal{G} \) of \( \mathcal{G} \) is a compact \( n \)-dim region, then:

\[ \int_{\partial \mathcal{G}} \omega = \int_{\mathcal{G}} \, d \omega \quad \text{for all} \; \omega \in \Lambda_{n-1}(M) \]

Definition: \( d \) is also called "co-boundary operator".
Remark:

- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H \text{dd} w = \int_h w = \int_{\partial H} w = 0$$

for geometric reasons, because, indeed, boundaries don’t possess boundaries:

- i.e.: Stokes implies $\partial^2 = 0 \Rightarrow \partial^2 = 0$

- Stokes links homology (geometric) to cohomology (algebraic).

Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes’ theorem is $\int_G df = \int_{\partial G}$, namely:

$$\int_a^b df = f|_a^b$$

(fund. thm of calculus)
Special case II: "Green's theorem."

\[ M = \mathbb{R}^2, \ G \subset \mathbb{R}^2 \ \text{a region} \]
with (closed) boundary curve \( \partial G \).

\[ \text{recall: this is automatic because } \partial G = 0 \]

Consider an arbitrary 1-form \( \omega \in \Lambda^1(M) \):

\[ \omega = \omega_1(x) \, dx^1 + \omega_2(x) \, dx^2 \]

Then:

\[ d\omega = d\omega_1(x) \wedge dx^1 + d\omega_2(x) \wedge dx^2 \]

\[ = \left( \frac{\partial \omega_1}{\partial x^1} \, dx^1 \wedge dx^2 \right) \wedge dx^1 \]
\[ + \left( \frac{\partial \omega_1}{\partial x^2} \, dx^1 \wedge dx^2 \right) \wedge dx^2 \]

\[ = \frac{\partial \omega_2}{\partial x^1} \, dx^2 \wedge dx^1 + \frac{\partial \omega_1}{\partial x^2} \, dx^1 \wedge dx^2 \]

\[ \implies d\omega = \left( \frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) \, dx^1 \wedge dx^2 \]

Now, Stoke's theorem \( \int_G d\omega = \int_{\partial G} \omega \) becomes:

\[ \int_G \left( \frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) \, dx^1 \, dx^2 = \int_{\partial G} (\omega_1 \, dx^1 + \omega_2 \, dx^2) \]

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for \( \partial G \), i.e., a differentiable map, invertible map \( \mathcal{D}:G \rightarrow \mathbb{R} \).
- Its inverse is a path: \( \gamma: J \subset \mathbb{R} \rightarrow \partial G \), with \( \gamma(t) = (x^1(t), x^2(t)) \)
- Now use \( dx^i = \frac{dx^i}{dt} \, dt \) to obtain an integral over \( J \subset \mathbb{R} \).
Special case of Gram's theorem:

Assume \( \omega \in \Lambda_1 \) is closed, i.e., \( d\omega = 0 \), i.e.,
\[
\frac{\partial \omega_1}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} = 0
\]

Then:
\[
\int_{\partial \Omega} \omega = 0
\]

Compare: (From the residue theorem)

If a function \( w: G \subset \mathbb{C} \to \mathbb{C} \) is holomorphic, i.e., it obeys the Cauchy-Riemann equations, then:
\[
\int_{\partial \Omega} w(z) \, dz = 0
\]

Indeed:

The Cauchy-Riemann equations mean that a differential form is closed and co-closed. We'll define "co-closedness" later.

Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for \( M = \mathbb{R}^3 \), namely
\[
\int_G \nabla \times \omega \, d\bar{G} = \int_{\text{1-dim boundary of } G} \omega \cdot d\bar{s}
\]

"Cross product": \( \vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \)

is indeed this special case:
\( \omega \in \Lambda_1(G) \) with \( \nabla \times \omega = d\omega \in \Lambda_2(G) \)
Before we can discuss the next example:

How to define the volume of a region \( G \subset M \) of a differentiable manifold \( M \)?

1. In \( \mathbb{R}^n \), we had:
   \[ V = \int \, dx^1 \cdots dx^n \]

2. In general, we need to choose a volume form
   \[ \Omega \in \Lambda_n \]
   among \( \Omega(p) \neq 0 \) \( \forall p \in G \). Then the (\( \Omega \)-dependent) volume
   is defined as:
   \[ V := \int_G \Omega \]

(proposed: \( G \) orientable \( \iff \exists \) volume forms \( \Omega \))

(in fact \( \infty \) many)

Special case III: Gauss' theorem

To obtain Gauss' theorem we need to define yet another derivative, the divergence of a vector field.

Recall: On \( \mathbb{R}^n \), the divergence of a vector field, \( \mathbf{F} \), was defined as

\[ \text{div} \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \mathbf{F}^i \]

How to generalize to arbitrary manifolds?

Where in this course did we see \( F_i^{(a)} \)? Before?

Recall:

\[ \left( L_{\xi_i} \zeta \right)_{j_1 \cdots j_n} \mathbf{F}(x) = \zeta_{j_1 \cdots j_n, k}(x) F^k(x) - \zeta_{j_1 \cdots j_n}^{k \cdots \ell}(x) F^k \xi_{\ell} \mathbf{F}(x) + \cdots \]
Strategy: If we choose \( \Omega \) to be the volume form, which on flat \( \mathbb{R}^n \) we may choose to be \( \Omega = dx^1 \cdots dx^n \), then the first term will drop out on \( \mathbb{R}^n \) b/c \( \partial_i \zeta_j = 0 \), and so we may be generalizing \( \zeta_{ji} \) on \( \mathbb{R}^n \)!

**Def:** The divergence of a vector field \( \zeta \) with respect to a volume form, \( \Omega \), is defined to be:

\[
\text{div}_\Omega \zeta := L^\zeta (\Omega)
\]

\( \Omega \): Lie derivative

\[ \]

- Assume \( \Omega = a(x) \sum_{i=1}^n dx^i \wedge \cdots \wedge dx^n \) (volume form)

  \( \zeta(x) = \sum_{i=1}^n \frac{\partial \zeta}{\partial x^i} (x) \) (vector field)

- Then:

\[
\text{div}_\Omega \zeta = L^\zeta (\Omega) = \zeta(a(x)) dx^1 \wedge \cdots \wedge dx^n + \ldots
\]

\[
+ \sum_{i=1}^n \frac{\partial \zeta}{\partial x^i} dx^1 \wedge \cdots \wedge L^\zeta (dx^i) \wedge \cdots \wedge dx^n
\]

(recall: \( L^\zeta (dx^i) = d(\zeta(x^i)) = d(\zeta^i) = \zeta^i \sum_{j=1}^n \frac{\partial \zeta^i}{\partial x^j} dx^j = \zeta^i dx^i \))

\[
\Rightarrow \text{div}_\Omega \zeta = \left( \sum_{i=1}^n \zeta^i \frac{\partial a}{\partial x^i} + \zeta^i \delta_{ji} \right) dx^1 \wedge \cdots \wedge dx^n
\]

\[
\Rightarrow \text{div}_\Omega \zeta = \frac{\partial}{\partial x^j} (\zeta^i \frac{\partial a}{\partial x^i}) \Omega
\]

Notation: If \( a(x) \equiv 1 \forall x \) then \( \text{div}_\Omega \zeta = \frac{\partial}{\partial x^j} \zeta^i \), as expected for the divergence in the simplest case.

Thus: Indeed, if \( a(x) \equiv 1 \forall x \) then \( \text{div}_\Omega \zeta = \zeta^i ; dx^1 \wedge \cdots \wedge dx^n \).
Now, we can derive Gauss' theorem from Stokes':

\[ \text{div}_{\Omega} \xi = L_{\xi} \Omega \in \Lambda_{m-1}(\Omega) \]

\[ \text{div}_{\Omega} \xi = (d \circ i_{\xi} + i_{\xi} \circ d) \Omega \]

\[ \Rightarrow \text{div}_{\Omega} \xi = d \circ i_{\xi} (\Omega) \]

We can now apply Stokes' theorem:

\[ \int_{\partial \Omega} d \xi = \int_{\Omega} i_{\xi} \]

\[ \int_{C} d i_{\xi} (\Omega) = \int_{\Omega} i_{\xi} (\Omega) \]

i.e.:

\[ \int_{\partial \Omega} \text{div}_{\Omega} \xi = \int_{\Omega} i_{\xi} (\Omega) \]  "Gauss' theorem"