How to describe the "shape" of a manifold?

Historically:

Example:

This motivates:

Recall:

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the non-triviality of the parallel transport of vectors on the manifold:

Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to $g$. Indeed, for $\xi_1 = \frac{2}{\delta x} \partial_1$, $\xi_2 = \frac{2}{\delta x} \partial_2$, we have $[\xi_1, \xi_2] = \delta_{12} \partial_3 = 0 \Rightarrow$ no shape info from $\xi_1$!
The Covariant Differentiation, $\nabla$:

**Definition:** Any linear map of tangent vector fields

$$\nabla : T'(M) \times T'(M) \to T'(M)$$

$$\nabla : \xi \otimes \eta \to \nabla_\xi \eta$$

obeying

(I) \hspace{1cm} \nabla_{\xi f} \eta = f \nabla_\xi \eta, \quad \forall f \in C^\infty(M)

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i}$$

(II) \hspace{1cm} \nabla_\xi (f \eta) = f \nabla_\xi \eta + \xi(f) \eta \quad \text{(Leibniz rule)}

is called a covariant derivative or affine connection.

**Note:**

For now, let us assume a metric has not (yet) been specified, so we are free to choose $\nabla$, and this choice defines the shape of $M$!

\[\nabla \text{ in a chart:} \]

- Choose or bases for $T_x(M)$, e.g.: \$\frac{\partial}{\partial x^i}\$
- Given a covariant derivative $\nabla$, consider its action on basis vectors, such as: \$\frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}\$

Recall: \$\frac{\partial^2}{\partial x^i \partial x^j} = 0\$

\[\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k}\]

The $\Gamma^k_{ij}$ are called "Christoffel symbols" or "connection coefficients".

Thus, via the axioms:

\[\nabla_\xi \eta = \nabla_{\frac{\partial}{\partial x^i}} (\xi^j \frac{\partial}{\partial x^j}) = \xi^j \nabla_{\frac{\partial}{\partial x^j}} (\xi^i \frac{\partial}{\partial x^i})\]

\[= \xi^j (\xi^i \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k} + \xi^k \frac{\partial}{\partial x^k} \Gamma^i_{jk}(x) \frac{\partial}{\partial x^j})\]

Note:

\[\eta^k_{,i} := \eta^k_{,i} + \eta^k \Gamma^i_{jk}(x)\frac{\partial}{\partial x^k}\]

Thus:

\[\nabla_\xi \eta = \xi^j \eta^k_{,j} \frac{\partial}{\partial x^k}\]
Important: the $\Gamma^k_{ij}$ transform non-tensorially when $x \to \tilde{x}$:

On one hand:

$$\nabla_{\partial \tilde{x}^k} \frac{\partial}{\partial \tilde{x}^a} = \Gamma^c_{ab} \frac{\partial}{\partial x^c} = \Gamma^c_{ab} \frac{\partial x^k}{\partial x^a} \frac{\partial}{\partial x^c} \tag{I}$$

On the other hand:

$$\nabla_{\partial x^k} \frac{\partial}{\partial x^a} = \Gamma^c_{ab} \left( \frac{\partial x^c}{\partial x^a} \frac{\partial}{\partial x^b} \right) \text{ use axiom (b) } \Rightarrow$$

$$= \Gamma^c_{ab} \frac{\partial x^c}{\partial x^a} \frac{\partial}{\partial x^b} \text{ use chain rule (c) } \Rightarrow$$

$$= \left( \frac{\partial x^c}{\partial x^a} \frac{\partial}{\partial x^b} \right) \frac{\partial x^k}{\partial x^c} + \frac{\partial x^k}{\partial x^a} \Gamma^c_{ij} \frac{\partial}{\partial x^c} \frac{\partial}{\partial x^b} \left( \frac{\partial x^k}{\partial x^a} \right) \tag{II}$$

(Compare I, II $\Rightarrow$

$$\Gamma^c_{ab} \frac{\partial x^k}{\partial x^a} = \frac{\partial^2 x^k}{\partial x^a \partial x^b} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \Gamma^k_{ij}$$

$\Rightarrow$

$$\Gamma^c_{ab} = \frac{\partial x^c}{\partial x^a} \frac{\partial^2 x^k}{\partial x^b \partial x^c} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^k}{\partial x^i \partial x^j} \Gamma^k_{ij}$$

This term is independent of $\Gamma^c_{ab}$ would be there, if the $\Gamma^c_{ij}$ were tensor coefficients in the $\frac{\partial}{\partial x^a}$, $\frac{\partial}{\partial x^b}$ bases.

$\Rightarrow$ $\Gamma^c_{ij}$ can be seen in one coordinate system and missing in another!

(Can be shown to be equivalent)

Physicists' definition of $\nabla$: Any set of $n^2$ functions $\Gamma^c_{ab} (x)$ which transform this way are defining a covariant derivative $\nabla$. 
The "absolute" covariant derivative $\nabla$:

Consider the covariant derivative but:
without choosing a direction vector $\xi$:

$$\nabla : T_x(M) \to T_x(M),$$

$$\nabla : \gamma = \gamma^i \frac{\partial}{\partial x^i} \to \nabla \gamma(x) = \gamma^i \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i} \quad (\text{if } \gamma \text{ with covariant } \xi),$$

Indeed: The contraction of $\nabla \gamma$ with $\xi$ yields:

$$\nabla \gamma(\xi) = \gamma^i \frac{\partial}{\partial x^i} \cdot \delta^j_{\xi} \cdot \frac{\partial}{\partial x^j} = \gamma^i \frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial x^i} = \nabla \gamma$$

We defined $\nabla$ algebraically. Now, extract the geometric meaning of $\nabla$:

Definition: Assume $\nabla$ is given. Choose a path $\gamma : \mathbb{R} \to M$.

Then, a tangent vector field $\gamma$ is called auto-parallel along $\gamma$, if

$$\nabla_{\gamma'} \gamma = 0$$

i.e. if $\gamma$ doesn't change under parallel transport along the path $\gamma$.

Note: We'll see that they always exist, i.e., we can always parallel transport a vector $\xi$ to a distance.
In a chart,
\[ \eta = \hat{\eta} \frac{\partial}{\partial x_i}, \]
and
\[ \gamma : [a, b] \to M, \]
\[ \gamma : t \to x^i(t) \]
and the tangent vector:
\[ \dot{x}^i(t) = \frac{dx}{dt} \frac{\partial}{\partial x^i} \]

**Thus:**
\[ \nabla_{\dot{x}} \eta = \frac{\partial}{\partial t} \left( \frac{\partial \eta^i}{\partial x^j} \right) = \frac{dx}{dt} \frac{\partial}{\partial x^i} \left( \frac{\partial \eta^i}{\partial x^j} \right) \]
\[ = \frac{d}{dt} \left( \frac{\partial x^i}{\partial x^j} \frac{\partial}{\partial x^i} + \eta^j \Gamma^i_{jk} \frac{\partial}{\partial x^k} \right) \]
\[ = \left( \frac{d}{dt} + \frac{dx}{dt} \frac{\partial}{\partial x^i} \right) \frac{\partial \eta^i}{\partial x^i} = 0 \]

\[ \Rightarrow \] \( \gamma \) autoparallel to \( \dot{x} \) means:
\[ \frac{d\eta^i}{dt} + \frac{dx^k}{dt} \Gamma^i_{kj} \eta^j = 0 \]
\[ \Rightarrow \] This is 1st order ODEs for \( \eta \). Thus:
Initial condition \( \gamma(\gamma(0)) \Rightarrow \) solution \( \gamma(\gamma(0)) \) exists at least locally

**Conclusion:**

Proposition:

Given a path \( \gamma : [t, s] \to M \), the autoparallel transport of a tangent vector \( \eta \) at \( \gamma(t) \) to \( \gamma(s) \) is unique.
\[ T(x) \Rightarrow T(x') \]
\[ \nabla_x \eta(x(t)) = \frac{d}{dt} \left|_{t=x} \right. \tau(s, t)(\eta(y_0(s)) \right) \]

Note: Since we can choose paths with arbitrary \( x \) and this equation can be used as a geometric definition of \( \nabla \).

\[ \nabla \text{ for arbitrary tensors:} \]

- The parallel transport map \( \tau(s, t) \) transports tangent vectors \( \eta \) from \( y_0(s) \) to \( y_0(t) \).

\[ \text{Definition: } \tau(s, t) \text{ also parallel transports the dual vectors } \omega, \text{ namely so that contraction is conserved:} \]

\[ \tau(\omega)(\tau(\eta)) = \omega(\eta) \quad (C') \]

- Extension of \( \tau \) to tensor products:

\[ \tau(s \otimes s') := \tau(s) \otimes \tau(s') \quad (T) \]

where \( s \) and \( s' \) are tensors of arbitrary rank.
\textbf{Definition:} 

\[ \nabla_{\bar{\gamma}} S(p) := \left. \nabla_{x} S'(x(t)) \right|_{t=0} \]

\[ = \left. \frac{d}{dt} \right|_{t=0} \tau(t,0)(S(x(t))) \]

here, \( \bar{\gamma} \) is any path through \( p \) obeying:

\[ \bar{\gamma}(0) = \xi(p), \quad \bar{\gamma}(0) = p \]

\textbf{Absolute covariant derivative:}

\[ \nabla_{S}(\eta_{1}, \ldots, \eta_{p}, \omega_{1}, \ldots, \omega_{q}, \xi) := \nabla_{\xi} S'(\eta_{1}, \ldots, \eta_{p}, \omega_{1}, \ldots, \omega_{q}) \]

\textbf{Properties of} \( \nabla \): 

\* \( \nabla \) is a derivation: (because \( \nabla \) inherits the Leibniz rule from \( \frac{d}{ds} \))

\[ \nabla_{S}(S \otimes S_{2}) = \frac{d}{ds} \left|_{s=0} \right. \tau(S \otimes \tau(S_{2})) = \frac{d}{ds} \left|_{s=0} \right. \tau(S) \otimes \tau(S_{2}) \]

\[ = \left[ \left. \frac{d}{ds} \right|_{s=0} \tau(S_{1}) \right] \otimes \tau(S_{2}) + \tau(S_{1}) \otimes \left. \frac{d}{ds} \right|_{s=0} \tau(S_{2}) \]

\[ = (\nabla_{S_{1}} S_{1}) \otimes S_{2} + S_{1} \otimes \nabla_{S_{2}} S_{2} \quad (A) \]

\* Eq. (C') implies that \( \nabla \) and contractions do commute.
Action of $\nabla$ on tensors in a chart?

- Recall: 
  \[ \nabla_\xi \frac{2}{\partial x^i} = \xi^k \Gamma^i_{kj} \frac{2}{\partial x^j} \]

- Problem: Find $\nabla_\xi dx^i = ?$

  \[ \Rightarrow \text{Consider } \eta \otimes \omega. \]

  \[ \Rightarrow \text{Differentiate: } \]

\[ \nabla_\xi (\eta \otimes \omega) = (\nabla_\xi \eta) \otimes \omega + \eta \otimes (\nabla_\xi \omega) \]

\[ \text{Contract:} \]

\[ (\text{by exercise above}) \]

\[ \psi(\omega(\xi)) = \omega(\nabla_\xi \eta) + (\nabla_\xi \omega)(\eta) \]

\[ \text{Scalar function} \]

\[ \Rightarrow \text{An expression for } \nabla_\xi (\omega(\xi)) = \psi(\omega(\xi)) - \omega(\nabla_\xi \eta) (\star) \]

Now: Choose $\omega := dx^i$ and $\eta := \frac{2}{\partial x^i}$

\[ \Rightarrow \left( \nabla_\xi dx^i \right) \left( \frac{2}{\partial x^i} \right) = \xi^k \left( \left< dx^i, \frac{2}{\partial x^i} \right> - \left< dx^i, \frac{2}{\partial x^i} \right> \right) = 0 \]

\[ = - \left< dx^i, \xi^k \Gamma^i_{kj} \frac{2}{\partial x^j} \right> = - \xi^0 \Gamma^i_{li} \]

\[ \Rightarrow \nabla_\xi dx^i = - \xi^0 \Gamma^i_{li} dx^i \]
For general tensors: (by exactly same strategy as above but applied to multiple tensor products, we obtain:)

\[ \nabla_i S' (\eta, \ldots, \eta, w, \ldots, w) \quad \text{(as in Eq. (\star) above)} \]

\[ = \xi(S'(\eta, \ldots, \eta, w, \ldots, w)) \]

\[ - S'(\nabla_i \eta, \eta, \ldots, w, \ldots, w) - \ldots \]

\[ - S'(\eta, \ldots, \eta, w, \ldots, w) \]

\[ - S'(\eta, \ldots, \eta, \nabla_i w, w, \ldots, w) + \ldots \]

\[ - S'(\eta, \ldots, \eta, w, \ldots, \nabla_i w) \]

Choosing the basis vectors \( dx^i \) and \( \frac{2}{\partial x^j} \) we obtain

for

\[ S = S^i_{\ldots,j} \frac{2}{\partial x^i} \theta \ldots \frac{2}{\partial x^j} \theta dx^i \theta \ldots \theta dx^j \]

that \( \nabla_i S' \) reads

\[ \nabla_i S' = \xi^{\alpha} S'^{i_{\ldots,j} \alpha} \frac{2}{\partial x^i} \theta \ldots \frac{2}{\partial x^j} \theta dx^i \theta \ldots \theta dx^j \]

with:

\[ S'^{i_{\ldots,j} \alpha} := S^{i_{\ldots,j} \alpha} + \Gamma^{i_{\ldots,j} \alpha} S^{i_{\ldots,j} \beta} \]

\[ + \ldots + \Gamma^{i_{\ldots,j} \alpha} S^{i_{\ldots,j} \beta} \]

\[ - \Gamma^{i \alpha} S^{i_{\ldots,j} \beta} \]

\[ - \ldots - \Gamma^{i \alpha} S^{i_{\ldots,j} \beta} \]
Special cases:

- Tangent vector fields:
  \[ \xi^j_{\;i} = \xi^j_{\;\kappa} + \xi^j_{\;\kappa} \Gamma_{\kappa i}^i \]

- Cotangent vector fields:
  \[ \omega^i_{\;jk} = \omega^i_{\;k\kappa} - \omega^i_{\;k\kappa} \Gamma_{\kappa i}^i \]

Recall: Specifying \( \Box \) specifies parallel transport of vectors and this should specify the manifold's shape, but how?

⇒ Indeed, \( \Box \) specifies Torsion & Curvature