Recall: So far, we have 2 ways to capture shape:

- Specified $g \Rightarrow$ specified distance in $\mathcal{M}$
  \[ \Rightarrow \text{implicitly specified "shape" of } \mathcal{M} \]

(Notice (fore essay): See also my new paper 1510.02725)

Then, new:

- Specified $\nabla \Rightarrow$ specified parallel transport in $\mathcal{M}$
  \[ \Rightarrow \text{implicitly specified "shape" of } \mathcal{M} \]

Question:

How does $\nabla$ determine "shape"? Through:

\[ \text{Torsion & Curvature!} \]

Recall:

\[ \Gamma^r_{ab} = \frac{\partial x^r}{\partial x^a} \frac{\partial x^b}{\partial x^c} \Gamma^c_{ij} + \frac{\partial x^r}{\partial x^c} \frac{\partial^2 x^a}{\partial x^i \partial x^j} \]

Notice: The antisymmetric part of $\Gamma$ transforms tensorially!

\[ \Gamma^k_{(ij)} := \frac{1}{2} \left( \Gamma^k_{ij} + \Gamma^k_{ji} \right) \]
\[ \Gamma^k_{(ij)} := \frac{1}{2} \left( \Gamma^k_{ij} - \Gamma^k_{ji} \right) \]
\[ \Gamma^k_{ij} = \Gamma^k_{(ij)} + \Gamma^k_{(ij)sym} \]
\[ \Rightarrow \Gamma^r_{(sym)ab} = \frac{\partial x^r}{\partial x^a} \frac{\partial x^b}{\partial x^c} \frac{\partial^2 x^a}{\partial x^i \partial x^j} \Gamma^k_{(sym)ij} \]

Definition: $\mathcal{T}^k_{ij} := 2 \Gamma^k_{(sym)ij}$ in the "Torsion tensor"

(Notice: Since $\Gamma$ is not a tensor, but $\Gamma_{(sym)}$ is, $\mathcal{T}$ is not a tensor)
In General Relativity, one assumes torsionless, i.e., \( T = 0 \).

Idea: "(Extended) equivalence principle:"

Christoffel \( \Gamma \) will express gravitational and pseudo forces. Therefore, we require that around each \( p \in M \) there exists a chart so that \( \Gamma^i(p) = 0 \) (i.e., no such forces in free fall).

This rules out the existence of torsion:

Why? The torsion is a tensor.

\[ \Rightarrow \text{\( \mathcal{T} \) transforms linearly with invertible Jacobian matrix} \]

\[ \overline{\mathcal{T}}_{jk}^i(x) = \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^l} \mathcal{T}_{kl}^i(x) \]

\[ \Rightarrow \text{If} \ \mathcal{T}_{ij} \ \text{vanishes in one chart, it vanishes in all charts.} \]

Proposition: Vice versa, if \( \mathcal{T}_{jk}^i(x) = 0 \) \( \forall x \in \mathcal{M} \), then there is for every \( p \in M \) a chart with \( \Gamma^i_{jk}(p) = 0 \):\n
Recall: \( \xi \) is autoparallel to a path \( \gamma : t \rightarrow x(t) \) if

\[ \nabla \xi_t = 0 \]

\( \nabla \) is parallel transported along the path \( \gamma \) in \( \mathcal{M} \).

Explicitly:

\[ \frac{d\xi^k}{dt} + \Gamma^k_{ij} \frac{dx^i}{dt} \xi^j = 0 \]

Geodesics: A curve \( \gamma : t \rightarrow x(t) \) is called a geodesic if \( \xi \) is autoparallel along \( \gamma \), i.e., if

\[ \nabla_{\dot{\gamma}} \gamma = 0 \]

Meaning: The path \( \gamma \) is in \( \mathcal{M} \) such that the path's tangent vectors are parallel translates of each other.
In charts, geodesics $x^\gamma(t)$ obey:

\[
\frac{d^2 x^\gamma}{dt^2} + \Gamma^\gamma_{\delta\mu}(x) \frac{dx^\delta}{dt} \frac{dx^\mu}{dt} = 0 \quad (\star)
\]

Theory of ordinary differential equations:

⇒ Given $p = x^\gamma(0)$, each initial condition $\xi = x^\gamma(0)$ belongs to a unique geodesic $x^\gamma$ of non-zero length.

Notice: If $x^\gamma(t)$ solves $(\star)$ then $x^\gamma(\lambda t)$ also solves $(\star)$ and for $\lambda \in \mathbb{R}$:

\[
x^\gamma_{\lambda \xi}(t) = x^\gamma(\lambda t) \quad (6)
\]

(Exercise: verify)

"Exponential map":

Consider a fixed point $p \in \mathcal{M}$.

The exponential map is defined through:

\[
\exp_p: T_p(M) \to \mathcal{M} \quad \text{((real)} \text{ly from a neighborhood of } p \text{ to a neighborhood of } p \text{ in } \mathcal{M})
\]

\[
\exp_p: \xi \to \exp_p(\xi) = x^\gamma(1)
\]

where $x^\gamma$ is the geodesic with $x^\gamma(0) = p, x^\gamma(0) = \xi$.

Observe:

From $(6)$ we obtain:

\[
x^\gamma(\lambda) = x^\gamma_{\lambda \xi}(1) = \exp_p(\lambda \xi) \quad (E)
\]
"Geodesic" or "Riemann normal" coordinates:

- \( \exp_p \) is a diffeomorphism from a neighborhood of \( 0 \in T_p(M) \cong \mathbb{R}^n \) into a neighborhood of the point \( p \in M \).

\[ \Rightarrow \exp_p \text{ provides a chart around } p: \]

- Choose a basis, say \( e_1, e_2, \ldots, e_n \) of \( T_p(M) \), then:
  \[ \xi = \xi^i e_i \]

- Through \( \exp_p \), the \( \xi^i \) become the coordinates of points in a neighborhood of \( p \in M \):
  \[ (\xi^1, \ldots, \xi^n) \rightarrow \exp_p (\xi^i e_i) \in M \]

- These \( \xi^i \) are called "normal" or "geodesic coordinates."

\[ \Rightarrow \text{Geodesics, } \gamma, \text{ through } p \text{ are straight lines in a normal c.s. about } p! \]

- Recall (E):
  \[ \gamma_p^\xi (\lambda) = \exp_p (\lambda \xi^i e_i) \]

  - For varying \( \lambda \), one moves along the geodesic in \( M \).

  - For varying \( \lambda \), one moves on a straight line in the coordinate system of the \( \xi^i \).

- Thus: In geodesic c.s., geodesics through \( p \) are straight lines of constant velocity \( \xi^i \).

- Does this mean \( \Gamma^k_{ij} (p) = 0 \)? No!
Geodesic eqn. at $p$:
\[
\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(p) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0
\]

Thus:
\[
\left( \Gamma^k_{ij}(p) + \Gamma^k_{ji}(p) \right) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0
\]

$\Rightarrow \Gamma^k_{ij}(p) = 0$ in geodesic coords.

$\Rightarrow$ Indeed: If the torsion vanishes, $\Gamma^k_{ij}(p) = \frac{1}{2} \mathcal{J}^k_{ij}(p) = 0$

then for each $p \in \mathcal{M}$ there exists a chart in which
the entire gravity and pseudo field vanishes at $p$:

Note:
Quantum fluctuations may induce torsion! So, let's nonetheless ask:

**What would torsion mean, geometrically?**

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**Abstract definition of Torsion:**

- Assume $\xi_1$ and $\xi_2$ are tangent vectors at $p \in \mathcal{M}$:

  Then, the **Torsion map** is defined as:

  $\mathcal{J} : T_p'(\mathcal{M}) \times T_p'(\mathcal{M}) \to T_p'\mathcal{M}$

  $\mathcal{J} : \xi_1, \xi_2 \to \mathcal{J} (\xi_1, \xi_2) := \nabla_{\xi_1} \xi_2 - \nabla_{\xi_2} \xi_1 - [\xi_1, \xi_2]$

- It is used to define the **Torsion tensor**, $\mathcal{J}$,

  $\mathcal{J} \in T_p'\mathcal{M}$

  through:

  $\mathcal{J} (\omega, \xi_1, \xi_2) := \langle \omega, \mathcal{J} (\xi_1, \xi_2) \rangle \in \mathbb{R}$
Choose canonical bases \( w := dx^k, g_1 := \frac{2}{\partial x^i}, g_2 := \frac{2}{\partial x^j} \):

\[
\mathcal{J}^k_{ij} := dx^k \left( \mathcal{J}(\frac{2}{\partial x^i}, \frac{2}{\partial x^j}) \right)
\]

\[
= \langle dx^k, \mathcal{J}(\frac{2}{\partial x^i}, \frac{2}{\partial x^j}) \rangle \quad \text{(in more convenient notation)}
\]

\[
= \langle dx^k, \frac{2}{\partial x^i}, \frac{2}{\partial x^j} \rangle - \left[ \frac{2}{\partial x^i}, \frac{2}{\partial x^j} \right] 
\]

\[
\frac{\partial}{\partial x^j} \left( \frac{2}{\partial x^i} \right) = \Gamma^r_{ij} \frac{2}{\partial x^r} 
\]

\[
= \langle dx^k, \Gamma^r_{ij} \frac{2}{\partial x^r} - \Gamma^r_{ji} \frac{2}{\partial x^r} \rangle = \Gamma^r_{ij} \delta^r - \Gamma^r_{ji} \delta^r
\]

\[
\implies \mathcal{J}^k_{ij} = \Gamma^r_{ij} - \Gamma^r_{ji}
\]

Geometric meaning of torsion? Parallelograms would not close!

Travel from \( p \) infinitesimally in \( \xi \) and then \( \eta \) direction, and compare with the reverse. \((\text{In flat space: } x^k + \xi^k + \eta^k = x^k + \xi^k + \eta^k)\)

Recall parallel transport: \( \nabla_{\xi} \gamma = 0 \)

\[
\frac{d\gamma^k}{dt} + \Gamma^k_{ij} \frac{dx^i}{dt} \gamma^j = 0
\]

Now use \( \gamma := \xi, \frac{dx}{dt} = \eta \):

\[
\tilde{\xi}(\tau) = ?
\]

\[
\tilde{\xi}^k(x + \tau \xi) \approx \xi^k(x) + \frac{d\xi^k}{d\xi}(x) \]

\[
= \xi^k(x) - \Gamma^k(x)_{ij} \eta^i \eta^j
\]

\[
\implies \text{Cds. of } \xi : \quad x^a + \eta^a + \xi^a - \Gamma^a(x)_{ij} \eta^i \eta^j
\]
Analogously obtain: \( \text{c.d.s. of } u: x^a + \xi^a + \eta^a - \Gamma^a_{ij} \xi^i \eta^j \)  
\[ \Gamma_{ij}^k \xi^i \eta^j \]

\( \Rightarrow \text{c.d. distance from } u \text{ to } \tilde{u} \text{ is: } \left( \Gamma_{ij}^k \xi^i - \Gamma_{ij}^k \xi^i \right) \eta^j \xi^j = T_{ij}^k \eta^j \xi^j \)

Comment: We had:

\[ \tilde{\xi}^k(x + \eta^i) \approx \xi^k(x^i) + \frac{\partial \xi^k}{\partial x^i}(x^i) = \xi^k(x^i) - \Gamma^k_{ij} \xi^i \eta^j \]

this is also:

\[ \xi^k(x^i) = (\eta^i \xi^k_{,i} + \Gamma^k_{ij} \eta^i \xi^j_{,i}) + \xi^k_{,i} \]

Thus: c.d. distance from \( u \) to \( \tilde{u} \) is:

\[ \left( x^a + \xi^a + \eta^a - \eta^i \xi^k_{,i} - \eta^i \xi^k_{,i} \right) - \left( \xi^k - \xi^k - \xi^k_{,k} - \eta^i \xi^k_{,i} - \eta^i \xi^k_{,i} \right) = T_{ij}^k \eta^j \xi^j \]

Recall that indeed: \( T: \eta, \xi \rightarrow T(\eta, \xi) = \nabla_\eta \xi - \nabla_\xi \eta - [\eta, \xi] \)

Curvature:

Assume \( \xi_1, \xi_2 \) and \( \xi_3 \) are tangent vectors at \( \text{p.e.m.} \)

1. The curvature map, \( R \), is defined through:

\[ R: \xi_1, \xi_2, \xi_3 \rightarrow R(\xi_1, \xi_2) \xi_3 = (\nabla_\xi_2 \xi_1 - \nabla_\xi_1 \xi_2 - \xi_1 [\xi_2, \xi_3]) \xi_3 \]

2. It defines the curvature tensor, \( R \),

\[ R \in T_3'(M) \]

through:

\[ R(\omega, \xi_1, \xi_2, \xi_3) := \langle \omega, R(\xi_1, \xi_2) \xi_3 \rangle \in \mathbb{R} \]
In a chart:

\[ R^i_{j;ks} = \langle dx^i, R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}\rangle \]

\[ = \langle dx^i, \left(\nabla_2 \nabla_2 \frac{\partial}{\partial x^j} - \nabla_2 \nabla_2 \frac{\partial}{\partial x^k} - \nabla_2 \nabla_2 \frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^i}\rangle \]

\[ = \langle dx^i, \nabla_2 \Gamma^i_{ks} \frac{\partial}{\partial x^j} - \nabla_2 \Gamma^i_{ks} \frac{\partial}{\partial x^l}\rangle \]

\[ = \langle dx^i, \left(\Gamma^i_{ksj} + \Gamma^i_{ksl} \right) \frac{\partial}{\partial x^j} - \Gamma^i_{ksj} \frac{\partial}{\partial x^l}\rangle \]

\[ = \Gamma^i_{kj} - \Gamma^i_{ksj} - \Gamma^i_{ksj} \frac{\partial}{\partial x^l}\]

(at origin of geodesic do they vanish.)

Curvature tensor's meaning?

Intuition:

- Contains derivatives of \( \Gamma \rightarrow \)

- Expresses variation in gravitational force \( \Rightarrow \)

- Expresses the strength and direction of "tidal force".

Geometry:

- Curvature expresses noncommutativity of two parallel transports, namely:
Proposition: (Ricci Identity)

Assume the torsion vanishes and that $\xi$ is a vector field. Then:

$$\xi^a_{\text{jcd}} - \xi^a_{\text{jdc}} = R^a_{\text{cdef}} \xi^b$$

(here: $\xi^a_{\text{jcd}} : = \xi^a_{\text{jcd}}$ etc.)

Remark: (too messy to derive, because need Taylor expansion)

Yes, e.g., test by Shouto Tsuchiya

It implies that for parallel transport along infinitesimal parallelogram:

$$(\xi^a - \xi^a) \approx \gamma^{bc} \nu^c R^a_{\text{cedf}} \xi^d$$

Proof of Ricci identity:

- Assume $\xi, \eta, \nu$ are vector fields.

- Then, $R(\xi, \eta)\nu := \nabla_{\xi} (\nabla_{\eta} \nu) - \nabla_{\eta} (\nabla_{\xi} \nu)$ reads

  $\nabla_{\xi} (\nabla_{\eta} \nu) - \nabla_{\eta} (\nabla_{\xi} \nu)$;

  in basis: $R^a_{\text{bed}} \xi^b \eta^c \nu^d = (\nabla_{\xi^d} \eta^c)_{\text{ijc}} \xi^i - (\nabla_{\xi^d} \xi^c)_{\text{ijc}} \eta^i$

- Terms cancel:

  $R^a_{\text{bed}} \xi^b \eta^c \nu^d = (\nabla_{\xi_d} \eta^c - \nabla_{\eta_d} \xi^c) \nu^d$

  $\Rightarrow \quad R^a_{\text{bed}} \xi^b \eta^c \nu^d = (\nabla_{\xi_d} \eta^c - \nabla_{\eta_d} \xi^c) \nu^d$

- True $\forall \xi, \eta \Rightarrow R^a_{\text{bed}} \nu^d = \nu^a_{\text{jcb}} - \nu^a_{\text{jdc}}$
The "Bianchi Identities":

- They are automatic relations among torsion and curvature, by construction.

- Preparation: \(\checkmark\) for maps!

Consider an arbitrary \(\mathcal{F}(M)\)-linear map:

\[ K: \xi_1 \times \xi_2 \times \ldots \times \xi_r \rightarrow \mathcal{F}(\xi_1, \ldots, \xi_r) \]

\(\text{target vector} \rightarrow \text{target vector}\)

i.e. at each \(p \in M\):

\[ K: T_p(M) \rightarrow T_p(M) \]

We can view \(K\) as a tensor \(\tilde{K} \in T_p(M)^\vee\), (as we did for \(R\) and \(J\))

namely:

\[ \tilde{K}(\omega, \xi_1, \ldots, \xi_r) = \langle \omega, K(\xi_1, \ldots, \xi_r) \rangle \]

Now let the usual derivative of the tensor \(K\) define the derivative of the map \(K\):

\[ \langle \omega, (\nabla_{\xi} K)(\xi_1, \ldots, \xi_r) \rangle := \nabla_{\xi} K(\omega, \xi_1, \ldots, \xi_r) \]

Using \(\nabla\) for map:
1st Bianchi Identity:

\[ \sum_{\text{cyclic}} R(\xi, \eta) v = \sum_{\text{cyclic}} \left( T(T(\xi, \eta), \nu) + (\xi \nabla) \right)(\eta, \nu) \]

2nd Bianchi Identity:

\[ \sum_{\text{cyclic}} \left( (\nabla_\xi R)(\eta, \nu) + R(T(\xi, \eta), \nu) \right) = 0 \]

with obvious simplification in case \( T = 0 \).

*Note:* They are automatically obeyed equations, just like any set of bin-operators obeys the Jacobi identity with respect to \([,] \). Indeed that's why:

**Proof of 1st Bianchi:** (assuming no torsion)

\[ \sum_{\text{cyclic}} R(\xi, \eta) v = 0 \]

Indeed:

\[ (\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi) v - \nabla_{[\xi, \eta]} v + \text{cyclic} \]

\[ = \nabla_\xi (\nabla_\eta v - \nabla_\nu \gamma) - \nabla_{[\gamma, \nu]} \gamma + \text{cyclic} \]

**Exercise:** Prove that, without torsion:\n
\[ \nabla_\gamma v - \nabla_\nu \gamma = [\gamma, v] \]

\[ = [\xi, [\gamma, v]] + \text{cyclic} \]

because again \([\alpha, [\beta, \gamma]] = [\alpha, \beta] \]

\[ = \left[ \xi, [\gamma, v] \right] + \text{cyclic} \]
\[ = 0 \quad \text{by Jacobi identity for all lin. maps.} \]

**Recall:**

Assume \( A, B, C \) are linear maps \( V \to V \)

Then: \([A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0\]

i.e., the Jacobi identity holds.

**Proof:** Simply spell out the commutators.

**Remark:** This means that, e.g., in quantum mechanics, all objects that need to be representable as operators on the Hilbert space must obey the Jacobi identity, e.g., generators of symmetries.