Assignment 10 - Fourier transforms
Exercise only, not to be handed in.

1. Show that the Fourier transform of

$$f(t) = e^{-|t|} \quad \text{is} \quad F(\omega) = \frac{2}{1+\omega^2}.$$

   Sketch the graphs in the time domain and in the frequency domain.

2. Show that the Fourier transform of T obeys:

$$\mathcal{F}(T(t)) = \left[\text{sinc }\left(\tfrac{1}{2}\omega\right)\right]^2,$$

   where $T(t)$ is the triangular gate function:

$$T(t) = \begin{cases} 1-|t|, & \text{if} \quad |t| < 1 \\ 0, & \text{if} \quad |t| > 1. \end{cases}$$

   Sketch the graphs of $T(t)$ and its Fourier transform.

3. Find the Fourier transform of

$$f(t) = \begin{cases} 1, & -1 < t < 0 \\ -1, & 0 < t < 1 \\ 0, & |t| > 1. \end{cases}$$

   in two ways,

   a) using the definition, and
   b) using the fact that $f(t) = T'(t)$, where $T(t)$ is the triangular gate function.

4. i) Knowing $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$, evaluate

$$I(\omega) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \cos \omega x dx.$$

   Hint: First, show that

$$I'(\omega) = -\omega I(\omega).$$

   and then solve this differential equation.

   ii) Use i) to show that the Fourier transform of a Gaussian is also a Gaussian:

$$\mathcal{F}(e^{-\frac{1}{2}x^2}) = \sqrt{2\pi}\, e^{-\frac{1}{2}\omega^2}.$$