Assignment 7 - Stokes's and Divergence theorem.

1. The surface integral of a scalar function can be given in a particularly simple form when the surface Σ is the graph of a function – *i.e.*, when its equation is of the type $z = f(x, y)$, where the domain D of f is the projection of the surface Σ on the xy-plane. In this special case, it is convenient to use x and y themselves as parameters, so that the surface $\vec{S}$ can be described through this map:

$$\vec{S}(x, y) = x\hat{\imath} + y\hat{\jmath} + f(x, y)\hat{k}.$$

   (a) Show that in this case, the surface integral becomes,

$$\iint_{\Sigma} \psi d\sigma = \iint_{D} \psi\left(x, y, f(x, y)\right) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy.$$

   (b) Use this formula to compute the surface area of the paraboloid $z = x^2 + y^2$ between $z = 0$ and $z = 2$.

   Hint: After setting up the integral in Cartesian coordinates, switch to polar coordinates and make the change of variable $1 + 4\rho^2 = u$.

   (c) The surface Σ is the piece of the plane $z + y = 1$ cut out by the cylinder $x^2 + y^2 = a^2$, oriented so that $\hat{n}$ points upwards. Calculate the flux through Σ of the vector field $\vec{F} = k(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$, where $k > 0$ is an arbitrary constant.

2. Evaluate

$$\iint_{\Sigma} \left[\vec{\nabla} \times \vec{F}\right] \cdot \hat{\mathbf{n}} d\sigma,$$

   where $\vec{F} = (y - z)\hat{\imath} - (x + z)\hat{\jmath} + (x + y)\hat{k}$, Σ is the portion of $z = 9 - x^2 - y^2$ with $z \geq 0$ and $\hat{\mathbf{n}}$ is the upward-pointing unit normal.

3. Use Gauss' theorem to evaluate

$$\iint_{\Sigma} \left(x^2 + y + z\right) d\sigma,$$

   where Σ is the surface of the ball $x^2 + y^2 + z^2 \leq 1$.

   Hint: To use the theorem, you first need to find a vector field $\vec{F}$ such that $\vec{F} \cdot \hat{\mathbf{n}} = x^2 + y + z$.