Assignment 9 - Fourier series.

1. The Fourier series for the function $f(x) = \frac{1}{2}x$, $-\pi < x < \pi$ is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n}.$$

   a) Sketch the graph of the function f_p which is the pointwise sum of the series on $\mathbb{R}$.

   b) Use Parseval's formula to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

   c) Show that

$$\tfrac{1}{12}(\pi^2 - 3x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nx}{n^2} \quad , \quad -\pi < x < \pi$$

   and

$$\tfrac{1}{12}x(\pi^2 - x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n^3} \quad , \quad -\pi < x < \pi$$

   d) Discuss whether the sum of each Fourier series in c) is piecewise continuous, continuous, piecewise C^1 or C^1 on $\mathbb{R}$. Sketch the graphs of the sum functions.

2. Use Parseval's formula and a suitable Fourier series to sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

   Hint: Look at Table 5.1. in the lecture notes.

3. Consider the function f defined by $f(x) = b, \quad 0 < x < L$.

   i) Extend f as an odd function and find its Fourier series (the sine series).

   ii) Discuss convergence of the series on $\mathbb{R}$, and sketch the graph of its sum function.

4. i) Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

   converges pointwise on $\mathbb{R}$, and uniformly on any finite interval.. Call its sum $f(x)$.

   ii) Obtain the Fourier series of the function g defined by

$$g(x) = \int_0^x f(t)dt,$$

   justifying your method by referring to appropriate theorems.

   iii) What can you say about continuity and smoothness of the function g? Explain.