where $k > 0$ is a constant. This law is plausible physically, since one expects that the larger is the spatial temperature gradient, represented by the gradient ∇u, the larger will be the rate of heat transfer. The constant k, called the *thermal conductivity*, depends on the material, being bigger for a good conductor of heat than for a poor conductor (e.g. $k_{\text{copper}}/k_{\text{glass}} \gg 1$).
□

Having given some physical examples of vector fields we now introduce the terminology formally.

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$. A *vector field on* $\mathcal{U}$ is a function $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$, whose domain is $\mathcal{U}$ and whose range is in $\mathbb{R}^n$, i.e. $\mathbf{F}$ assigns to each $\mathbf{x} \in \mathcal{U}$ a unique vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$.

The vector fields we work with will usually be *of class* C^1, which means that the n component functions,

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}))$$

are C^1 functions.

We shall also assume that the domain $\mathcal{U}$ is a *connected set*, which means that $\mathcal{U}$ is not the union of two or more disjoint sets.

1.2.2 Field lines of a vector field

One visualizes a vector field $\mathbf{F}$ on an open set $\mathcal{U} \subset \mathbb{R}^3$ as a "field of vectors", represented by arrows, attached to the points of $\mathcal{U}$. The *length* of the vector at a point gives the *strength of the field* at the point, and the *arrow* gives *the direction of the field*.

It is also helpful to think of the family of curves in $\mathcal{U}$ with the property that at each point P *the tangent to the curve through* P *equals the vector field evaluated at* P. These curves are called the *field lines*[1] *of the vector field*. One thinks of a field line threading its way through the vector field, always following the direction of the vector field (see Figure 1.18).

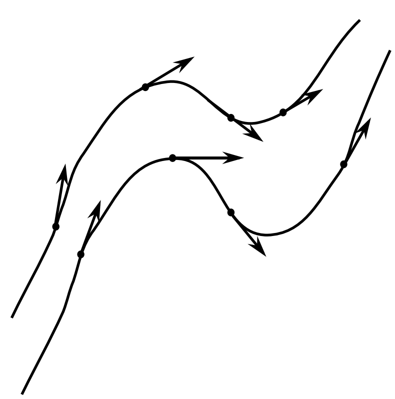

Figure 1.18: Two field lines of a vector field.

[1]These curves are also called *integral curves* of the vector field.

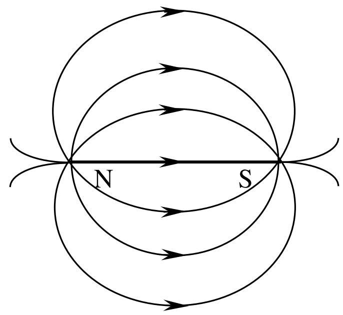

Figure 1.19: Field lines of a magnetic field.

In the case of the *velocity field of a fluid* the field lines are simply the paths of the fluid particles (see Figures 1.16 and 1.18). In the case of a magnetic field one can do a simple experiment to visualize the field lines. Take a sheet of paper and sprinkle it with iron filings. Then put a bar magnet under the paper and shake the paper slightly. You will observe the iron filings arranging themselves in lines going from one magnetic pole to the other. The strength of the magnetic field is revealed by the density of packing of the filings. In this way one obtains a picture of the field lines of the magnetic field.

We now consider the problem of determining the field lines of a given vector field $F(\mathbf{x})$.

Let the curve $\mathbf{x} = \mathbf{g}(t)$, assumed C^1, be a field line. Its tangent vector is $\mathbf{g}'(t)$, and thus the defining condition of a field line is written

$$\mathbf{g}'(t) = \mathbf{F}(\mathbf{g}(t)). \tag{1.31}$$

(see Figure 1.20). Equation (1.31) is a differential equation for the unknown vector-valued function $\mathbf{g}(t)$. If you want to find the field line through a given point $\mathbf{x}_0$ then you should impose the initial condition

$$\mathbf{g}(t_0) = \mathbf{x}_0. \tag{1.32}$$

DEs such as (1.31) are studied in depth in the course AM 451. In general they can only be solved numerically using a computer. For this course, however, it will be enough to concentrate on simple types that can be solved explicitly, as in the examples to follow.

Example 1.12:

Find the field lines of the vector field

$$\mathbf{F}(x, y) = (-y, x) \tag{1.33}$$

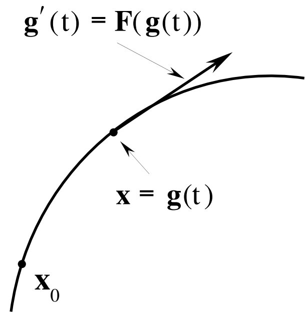

Figure 1.20: The field line of a vector field $\mathbf{F}$ through a given point $\mathbf{x}_0$.

in $\mathbb{R}^2$, and sketch the field portrait.

Solution: A field line $\mathbf{x} = \mathbf{g}(t)$ satisfies

$$\mathbf{g}'(t) = \mathbf{F}(\mathbf{g}(t)).$$

In terms of components $\mathbf{g}(t) = (x(t), y(t))$, this reads

$$(x'(t), y'(t)) = (-y(t), x(t)),$$

giving

$$\frac{dx}{dt} = -y(t), \quad \frac{dy}{dt} = x(t). \tag{1.34}$$

These two coupled DEs can be written as a single DE by using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

giving

$$\frac{dy}{dx} = -\frac{x}{y}$$

by equations (1.34). Solving this separable DE yields

$$\int y\,dy = -\int x\,dx,$$

giving

$$x^2 + y^2 = C,$$

where C is a constant.

The conclusion is that *any field line of the given vector field is a circle centred on the origin*. Equations (1.34) show that the circles are traversed counterclockwise, giving Figure 1.21. We note that the field line through a given point (x_0, y_0) is the circle of radius $\sqrt{x_0^2 + y_0^2}$ – specifying a point on the field line fixes the value of the constant C. $\square$

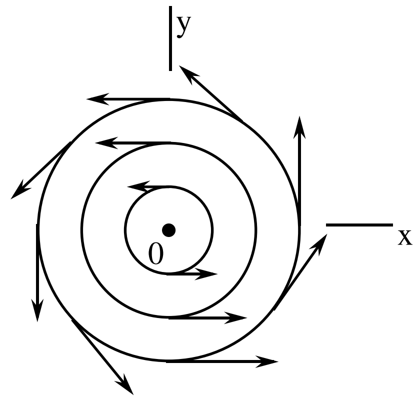

Figure 1.21: The field lines $x^2 + y^2 = C$ of the vector field $\mathbf{F} = (-y, x)$.

Exercise 1.3:

The vector field in Example 1.12 can be interpreted physically as the velocity field of a rigidly rotating disc with unit angular velocity. Verify that

$$\theta' \equiv \frac{d\theta}{dt} = 1.$$

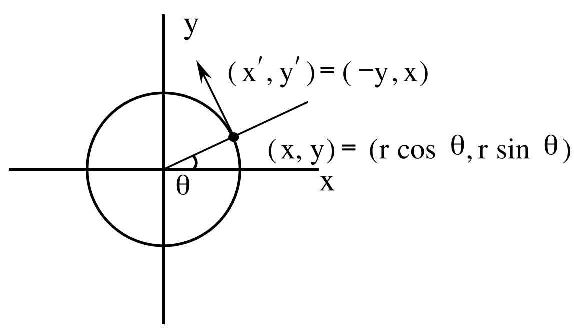

Figure 1.22: Velocity of a point on a rotating disc.

Example 1.13:

Find the field lines of the vector field

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \tag{1.35}$$

on the open set $\mathcal{U} = \mathbb{R}^2 - \{(0, 0)\}$, and sketch the "portrait".

Solution: We have to solve the DEs

$$\frac{dx}{dt} = \frac{-y}{x^2+y^2}, \quad \frac{dy}{dt} = \frac{x}{x^2+y^2}.$$

Proceeding as in Example 1.12, these equations lead to the same DE

$$\frac{dy}{dx} = -\frac{x}{y},$$

giving

$$x^2 + y^2 = C,$$

where C is a constant, as the field lines. $\square$

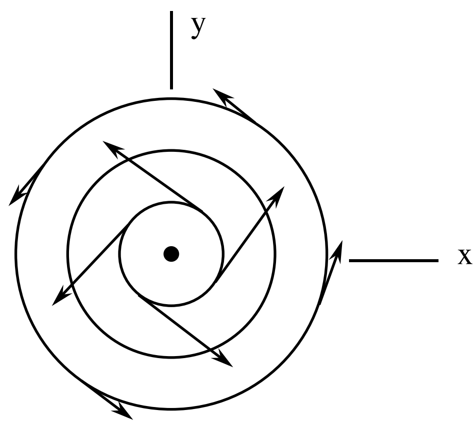

Figure 1.23: Field lines of the vector field (1.35).

Comment:

The vector field (1.35) could represent the velocity field of a fluid swirling down a drain. Note that for Example 1.12,

$$\| \mathbf{F}(x,y) \| = \sqrt{x^2+y^2},$$

i.e. the speed equals the distance from the origin, while for Example 1.13,

$$\| \mathbf{F}(x,y) \| = \frac{1}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0),$$

i.e. the speed equals the reciprocal of the distance from the origin. We have indicated this difference in Figures 1.21 and 1.23 by the size of the arrows. We note that Examples 1.12 and 1.13 illustrate that *different vector fields can have the same field lines.*

Exercise 1.4:

Find the field lines of the vector field $\mathbf{F}(x,y) = (x, 2y)$ in $\mathbb{R}^2$, and sketch the field portrait.

Gradient vector fields:

The field lines of a vector field $\mathbf{F}(\mathbf{x}) = \nabla u(\mathbf{x})$ in $\mathbb{R}^2$ that is the gradient of a scalar field can be drawn without solving a DE. We know (Calculus 3) that the gradient ∇u of a scalar field u is orthogonal to the level curves $u =$ constant of the scalar field. It follows that *the field lines of the vector field $\mathbf{F} = \nabla u$ are the orthogonal trajectories of the family of level curves of u.*

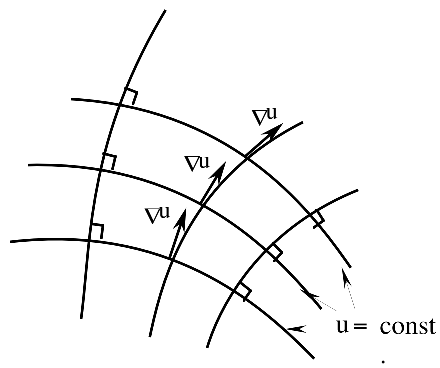

Figure 1.24: The field lines of $\mathbf{F} = \nabla u$ intersect the level curves $u =$ constant orthogonally.