Homework 2 - Paths and curves.

1. Find a path in $\mathbb{R}^2$ whose image set is:
   (a) The circle with radius 4 and center at $(-1, 3)$.
   (b) The ellipse which in Cartesian coordinates has the equation,
   $$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$
   (c) The branch of the hyperbola,
   $$\frac{x^2}{16} - \frac{y^2}{9} = 1$$
   lying in the lower-left quadrant $(x, y) \leq 0$.

2. A point moves along a right-circular helix C, which is the curve of the path $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$ given by,
   $$\vec{\gamma}(t) = \cos t\hat{\imath} + \sin t\hat{\jmath} + 2t\hat{k}.$$
   (a) Sketch C.
   (b) Find the velocity and the speed of the point.
   (c) Find the acceleration and comment on its direction.

3. Given that the Euclidean norm is defined as $||\vec{x}||^2 = \vec{x} \cdot \vec{x}$ and that $\vec{x}(t)$ is a differentiable function, show that
   (a) $\left(||\vec{x}(t)||^2\right)' = 2\,\vec{x}(t) \cdot \vec{x}\,'(t)$.
   (b) $(||\vec{x}(t)||)' = \vec{x}(t) \cdot \vec{x}\,'(t)/||\vec{x}||$

4. Find the field lines and sketch the field portrait for the following vector fields $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$.
   (a) $\vec{F}(x, y) = x\hat{\imath} + x^2\hat{\jmath}$.
   (b) $\vec{F}(x, y) = x^2\hat{\imath} + x\hat{\jmath}$.
   (c) $\vec{F}(x, y) = y\hat{\imath} + y^2\hat{\jmath}$.

5. A fluid flow in $\mathbb{R}^3$ has velocity field $\vec{v} = -x^2\hat{\imath} + x\hat{\jmath} + 2z\hat{k}$.
   (a) Find the equation of the field line which passes through an arbitrary point (x_0, y_0, z_0) at time $t = 0$.
   (b) A fluid particle is at position $(1, 0, e^2)$ at time $t = 0$. Find its position at time $t = 1$.

6. Sketch the two curves $\vec{\gamma}(t) = t\hat{\imath} + t^2\hat{\jmath}$; $t \in [1, 5]$ and $\vec{\beta}(\tau) = e^\tau\hat{\imath} + e^{2\tau}\hat{\jmath}$; $\tau \in [0, \ln 5]$.
   How are they the same? How are they different?

7. A curve joining the points P_1 and P_2 in $\mathbb{R}^2$ is parameterized by two different paths,
   $$\vec{\gamma}(t); t_1 \leq t \leq t_2 \quad \text{and} \quad \vec{\beta}(\tau); \tau_1 \leq \tau \leq \tau_2,$$
   where $\tau = h(t)$ and $h'(t) > 0$. Assume that $\vec{\gamma}$, $\vec{\beta}$, and h are all C^1.
   (a) Prove that
   $$\int_{t_1}^{t_2} ||\vec{\gamma}'(t)||\, dt = \int_{\tau_1}^{\tau_2} ||\vec{\beta}'(\tau)||\, d\tau.$$
   (b) Interpret the result geometrically.