Homework 3 - Paths integrals and gradient fields.

1. A force field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\vec{F}(x, y, z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Compute the work $\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$ done by $\vec{F}$ in moving a particle along the parabola $z = 0$, $y = x^2$, from $x = -1$ to $x = 3$.

2. A vector field is given by $\vec{F}(x, y, z) = x^2\hat{\imath} + xy\hat{\jmath} + 1\hat{k}$.
   Compute $\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$ where the curve $\mathcal{C}$ parameterized by $\vec{\gamma} : [0, 1] \to \mathbb{R}^3$ is given as $\vec{\gamma}(t) = t\hat{\imath} + t^2\hat{\jmath} + 1\hat{k}$.

3. Find the length of the curve parameterized by $\vec{\gamma}(t) = \cos t\hat{\imath} + \sin t\hat{\jmath} + t\hat{k}$; $\quad \pi \le t \le 3\pi$.

4. A fence is built with posts arranged in the xy-plane along the quarter circle $x^2 + y^2 = 25$, $(x, y) > 0$.
   The height of the fence at point (x, y) is given by $h(x, y) = 10 - x - y$ feet.
   (a) Calculate the area of one side of the fence.
   (b) Calculate the average height of the fence above the xy-plane.

5. (a) Suppose that $\vec{g} : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and that $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field. Define the composition $h : \mathbb{R} \to \mathbb{R}$ by,
   $$h(t) = f\left(\vec{g}(t)\right).$$
   Show that if f and $\vec{g}$ are C^1 functions, then
   $$h'(t) = \nabla f\left(\vec{g}(t)\right) \cdot \vec{g}\,'(t).$$
   (b) Suppose a vector field $\vec{F}$ is given by the gradient of a scalar field Φ according to $\vec{F} = -\vec{\nabla}\Phi$. Suppose also that $\vec{\sigma}(t)$ is a field line of $\vec{F}$. Show that $\Phi(\vec{\sigma}(t))$ is a decreasing function of t.
   (c) If a particle of mass m moves in a gradient force field $\vec{F} = -\nabla V$, the total energy of the particle is given by,
   $$E = \frac{1}{2}m||\vec{v}(t)||^2 + V(\vec{x}(t)),$$
   where $\vec{x}(t)$ is the particle's position at time t and $\vec{v}(t) = \vec{x}'(t)$ is its velocity. Assume that Newton's second law,
   $$\vec{F} = m\vec{a}(t),$$
   is satisfied, where $\vec{a}(t) = \vec{v}'(t)$ is the particle's acceleration at time t. Prove that the energy is conserved, *i.e.*, that $\frac{dE}{dt} = 0$. That is why a gradient force field is called sometimes called a *conservative* force field.