Assignment 4 - Conservative fields and Green's theorem.

1. A force $\vec{F}$ acts on a particle that is moving in the plane along the semi-circle parameterized by $\vec{\gamma}(t) = (-\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$; $t \in [0, \pi]$. Find the work $W = \int_C \vec{F} \cdot d\vec{s}$ done by the force field when

   (a) $\vec{F} = \sqrt{x^2 + y^2}\ \hat{\imath}$

   (b) $\vec{F} = \sqrt{x^2 + y^2}\ \hat{\tau}$, where $\hat{\tau}$ is the unit vector *tangential* to the path.

2. Evaluate $\oint_C \vec{F} \cdot d\vec{s}$, where $\vec{F} = 2x(x + y)\hat{\imath} + (x^2 + xy + y^2)\hat{\jmath}$ and the positively-oriented curve C encloses the square with vertices $(0, 0), (1, 0), (1, 1)$ and $(0, 1)$.

3. Let D be the upper half-plane ($y > 0$) and consider the vector field $\vec{F} : D \mapsto \mathbb{R}^2$ given by $\vec{F}(x, y) = \left(\frac{7}{y}, \frac{-7x}{y^2}\right)$. Test whether $\vec{F}$ is a gradient field, and if so, find a potential Φ.

4. Consider the closed curve C, which is the image of the path $\vec{\gamma}(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$.

   (a) Given the velocity field $\vec{v} = (-y, x)$, sketch the velocity vector at various points along C and predict whether the circulation of $\vec{v}$ around C is positive, negative or zero. Verify your prediction.

   (b) Repeat 4a with $\vec{v} = (x, y)$.

5. In Newton's theory of gravitation, the Earth's gravitational field is given by,

$$\vec{g}(\vec{r}) = \frac{-GM}{r^3}\, \vec{r},$$

   where $\vec{r} = (x, y, z)$ is the position vector relative to the centre of the Earth, $r = ||\vec{r}||$, M is the mass of the Earth and G is the gravitational constant. The force acting on a particle of mass m at position $\vec{r}$ is given by $\vec{F} = m\vec{g}(\vec{r})$.

   (a) A rocket of mass m is launched from position $(0, 0, R)$ and travels to $(0, 0, R + h)$ in a time T, along the straight line path C_1. A similar rocket travels along the spiral path C_2 given by

$$\vec{r} = \vec{h}(t) = \left(R + \frac{h}{T}\, t\right)\left(-\sin\frac{2\pi t}{T}, 0, \cos\frac{2\pi t}{T}\right),$$

   with $0 \le t \le T$. By evaluating the appropriate line integrals, show that the work done by the gravitational field along C_1 equals the work done along C_2 (Figure 1).

   (b) Show that the gravitational field is conservative by finding a potential Φ. Hence, verify the expression for work obtained in 5a.

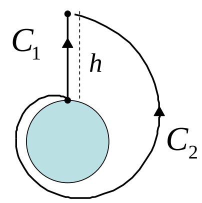

Figure 1: For question 5. The origin is at the centre of the Earth.