Assignment 5 - Gradients, Circulation, Surfaces, Surface Integrals.

1. In the plane, let D be a simply-connected region whose boundary ∂D is a piecewise C^1, simple, closed curve, oriented counterclockwise. Let $\hat{n}$ be the outward unit normal vector to D. Given two functions, $f(x,y)$ and $g(x,y)$, both C^2 on an open set containing the domain D, show that:

   (a)
$$\oint_C \left(f\vec{\nabla}g\right)\cdot d\vec{s} = -\oint_C \left(g\vec{\nabla}f\right)\cdot d\vec{s},$$
   for any piece-wise C^1 *closed* curve C in the domain D.

   (b)
$$\iint_D \left(f\nabla^2 g\right)dxdy = \oint_{\partial D}\left(f\vec{\nabla}g\right)\cdot\hat{n}\,ds - \iint_D\left(\vec{\nabla}f\cdot\vec{\nabla}g\right)dxdy$$
   where $\nabla^2 g = \vec{\nabla}\cdot\vec{\nabla}g = g_{xx}+g_{yy}$.

   (c)
$$\iint_D \left(f\nabla^2 g - g\nabla^2 f\right)dxdy = \oint_{\partial D}\left(f\vec{\nabla}g - g\vec{\nabla}f\right)\cdot\hat{n}\,ds$$

2. Calculate the circulation along the given bounding curve and vector field.

   (a) The circle of radius a centered at the origin, and the vector field $\vec{F} = -x\hat{\imath} + y\hat{\jmath}$.

   (b) The square with vertices $(0,0),(1,0),(1,1)$ and $(0,1)$, and the vector field $\vec{F} = 3x^2y\hat{\imath} + x^3\hat{\jmath}$.

   (c) The ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and the vector field $\vec{F} = xy^2\hat{\imath} + 2x^2y\hat{\jmath}$.

3. Sketch the following surfaces $\vec{S}$. For each point on the surface, find two linearly independent tangent vectors and write down an equation for the tangent plane in normal form.

   (a) $\vec{S} = (\cos v\sin u)\hat{\imath} + (1+\sin v\sin u)\hat{\jmath} + (\cos u)\hat{k}$, $(u,v)\in[0,\pi]\times[0,2\pi]$

   (b) $\vec{S} = (\sin v)\hat{\imath} + (u)\hat{\jmath} + (\cos v)\hat{k}$, $(u,v)\in[-1,3]\times[-\frac{\pi}{2},\frac{\pi}{2}]$

4. Evaluate the surface integral $\iint_\Sigma f\,dS$ where,

   (a) $f(x,y,z) = xz$ and Σ is the boundary of the region D in $\mathbb{R}^3$ inside the cylinder $x^2+y^2=1$ between the planes $z=0$ and $z=x+2$.

   (b) $f(x,y,z) = x^2$ and Σ is the boundary of the region D in $\mathbb{R}^3$ inside the cone $z^2=x^2+y^2$ and between the planes $z=1$ and $z=2$.