Assignment 6 - Vector Fields, Surfaces, Surface Integrals, ∇.

1. Suppose that $\mathbf{v} = (0, y, 0)$ is the velocity field of a fluid and let S be the surface defined by $y = x^2 + z^2$, with $y \leq 1$. Calculate the volume of fluid crossing S in unit time, in the direction of increasing y.

2. Verify the following properties of the curl.

   (i) *Sum* of two vector fields:
   $$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}\,.$$

   (ii) *Product* of a scalar field and a vector field:
   $$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}\,.$$

   (iii) *Vector product* of two vector fields:
   $$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

3. Verify the following "zero identities":

   (i) $\nabla \times (\nabla f) = \mathbf{0}$, for any C^2 scalar field f.

   (ii) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, for any C^2 vector field $\mathbf{F}$.

4. Let f, g be C^2 scalar fields on $\mathbb{R}^3$.

   (i) Verify that $\nabla \times (f\nabla g) = \nabla f \times \nabla g$.

   (ii) Simplify $\nabla \times (f\nabla g + g\nabla f)$.

   (iii) Verify that $\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.

5. Verify the "curl of a curl" identity
   $$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}.$$
   where ∇^2 is the Laplacian operator.

6. Use Maxwell's equations (see Section 4.1.1) to show that if the charge density field ε and the current density field $\mathbf{J}$ are zero, the electric vector field $\mathbf{E}$ satisfies the "wave equation"
   $$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} = 0.$$
   This is the fundamental equation that describes the propagation of electromagnetic waves, such as light.