Assignment 8 - Fourier series.

1. Consider the function f defined by

$$f(x) = | \sin x |, \quad \text{for} \quad -\pi < x < \pi.$$

   a) Find the Fourier series of f.

   b) Sketch the graph of the function to which the series converges pointwise on $\mathbb{R}$. Use the pointwise convergence theorem to justify your answer.

   c) By choosing a suitable value of x in the series that you have found, show that $\sum_{k=1}^{\infty} \dfrac{1}{4k^2 - 1} = \frac{1}{2}$.

2. For each of the following functions defined on the interval $(0, \pi)$:

   (a) Find the Fourier sine series.

   (b) Find the Fourier cosine series.

   (c) Sketch the functions to which the Fourier sine and cosine series converge on $\mathbb{R}$.

   (i) $f(x) = 2x - \pi$

   (ii) $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2 \\ -1 & \text{if} \quad \pi/2 < x < \pi \end{cases}$

   (iii) $f(x) = \sin x$.

3. (a) Show that the Fourier series for $f(x) = x^2$, $-\pi < x < \pi$, is

$$\frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(nx)}{n^2}.$$

   (b) Use the pointwise convergence theorem to find the pointwise sum function of the series on $\mathbb{R}$. Sketch its graph.

   (c) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (This is the Basel problem, which was famously first solved by Euler long before Fourier methods were developed)

   (d) Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

4. Let f be a C^1 function on $[-\pi, \pi]$. Prove that the Fourier coefficients of f satisfy

$$| a_n | \leq \frac{K}{n}, \quad | b_n | \leq \frac{L}{n}, \quad n = 1, 2, \cdots,$$

   for some constants K and L.