Faculty of Mathematics
University of Waterloo
AMATH473/673, PHYS454 Advanced Quantum Mechanics
Final Exam - Fall Term 2020

Time: 12:00pm-1:40pm + 40min for upload to Crowdmark **Date:** December 18, 2020.

Important:

(A) The solutions need to be handwritten on paper. Each page needs to be uploaded to Crowdmark twice, once plain (same as for homework), and once as a selfie with your Watcard or other official ID with picture and name. Example pictures are linked to on the course's home page. It's the same procedure as for the midterm.

(B) You are allowed to use the lecture notes for this exam. You are not allowed to communicate with others during the exam. Except, you can communicate with the instructor via email if a question is unclear.

(C) If a problem asks to give nontrivial examples then these must be examples that you yourself come up with. This means that your examples cannot be examples from textbooks and exercises or any other source that is not you. There is no need to make the examples extra complicated. But the examples must be nontrivial enough so that it is very unlikely for someone else to come up with the same example.

(D) I am very sorry to have to say this, and of course this concern doesn't apply to most of you, but plagiarism is a very serious matter because exams absolutely have to be fair for all. Therefore, I have to tell you that when a case of plagiarism is found, then I really have no choice. As a prof, I am obliged to report each and every case to the Associate Dean. Folks, please spare me having to do that! In cases where plagiarism is only suspected but not clear, the involved students may have to do an oral exam online.

(E) Your answers must be stated in a clear and logical form in order to receive full marks.

1. Assume $\alpha \in [1, 2]$. Which of the following operators are unitary or self-adjoint or both or neither? Just state the property, no proof:

(4) 1a. $\hat{x} + \alpha\hat{p}$

(4) 1b. $e^{i\alpha\hat{x}}$

(4) 1c. $e^{i\alpha\hat{x}\hat{p}}$

(4) 1d. $(\hat{p} - \alpha i 1)(\hat{p} + \alpha i 1)^{-1}$

(4) 1e. $|a\rangle\langle a| + e^{i\alpha}|b\rangle\langle b|$ (assume that $|a\rangle$, $|b\rangle$ form an orthonormal basis)

(4) 2a. Give an example of two different quantum mechanical Hamiltonians that would be the same in classical mechanics.

(4) 2b. Explain why the operators $\hat{x}$ and $\hat{p}$ cannot possess finite-dimensional Hilbert space representations.

(4) 2c. The swap of two systems amounts to a rotation by how many degrees?

(4) 2d. In an infinite-dimensional Hilbert space many operators possess a domain that is smaller than the Hilbert space, i.e., they cannot be defined to act on all Hilbert space vectors because they would map some vectors to the outside of the Hilbert space. In a discrete basis, give a nontrivial example of such an operator and a Hilbert space vector that it would map to the outside of the Hilbert space.

(5) 2e. Assume $\hat{f}$ is an observable and $\hat{U}$ is unitary. Show that $\hat{g} := \hat{U}^\dagger \hat{f} \hat{U}$ and $\hat{f}$ possess the same eigenvalues.

(4) 3a. When is a density operator $\hat{\rho}$ describing a pure state?

(4) 3b. Write down a nontrivial 2×2 density matrix (i.e., a self-adjoint 2×2 matrix) and calculate its purity.

(4) 3c. Now consider a composite system AB. Assume that it is in a pure state, $|\Omega\rangle$. Under which condition is $|\Omega\rangle$ called entangled?

(4) 3d. In a composite system AB that is in a pure state, explain, *qualitatively,* how the amount of purity of subsystem A is related to the amount of entanglement between the two subsystems A and B.

(4) 3e. Explain how the entanglement entropy of a subsystem is defined.

4. Consider a composite system AB which is in this state (with $|E_1\rangle, |E_2\rangle$ orthonormal):

$$|\psi\rangle = \frac{1}{\sqrt{3}}\ |E_1\rangle \otimes |E_1\rangle + \sqrt{\frac{2}{3}}\ |E_2\rangle \otimes |E_2\rangle)$$

(4) 4a. Is the state $|\psi\rangle$ a pure state?

(4) 4b. Explain whether or not the state $|\psi\rangle$ is entangled.

(5) 4c. i) Calculate the density matrix of the subsystem A (whose Hilbert space is the first tensor factor).
ii) Is subsystem A in a pure or in a mixed state?

5. Consider a quantum system that consists of two *identical bosonic* subsystems. The subsystems each possess a two-dimensional Hilbert space with basis $|E_n\rangle$, with $n \in \{1,2\}$ and Hamiltonian $\hat{H} = E_1|E_1\rangle\langle E_1| + E_2|E_2\rangle\langle E_2|$ with $E_2 > E_1$. The total Hamiltonian is given by $\hat{H}_{total} = \hat{H} \otimes 1 + 1 \otimes \hat{H}$. The system is exposed to a heat bath at temperature T.

(4) 5a. What is the eigenbasis of H_{total} and what are its eigenvalues?

(4) 5b. What is the density matrix, $\hat{\rho}_{total}$, of the system in the energy eigenbasis?

(5) 5c. i) As the temperature is raised towards infinity, what is the probability for finding the system in the ground state $|E_1\rangle|E_1\rangle$? ii) In comparison, what would be that probability (for finding the system in the ground state at very high temperatures) if the two subsystems where distinguishable?

(4) 6a. Consider a composite system AB whose two *non-identical* subsystems A and B start at time $t = 0$ unentangled and in the pure states $|\phi\rangle$ and $|\psi\rangle$ respectively. Assume that the total Schrödinger Hamiltonian is of the form

$$\hat{H} = \hat{H}_A \otimes 1 + 1 \otimes \hat{H}_B + \hat{H}_{int}$$

with the interaction Hamiltonian being of the form $\hat{H}_{int} = \hat{f} \otimes \hat{g}$. If nothing else is known about the states and the operators, should we expect that, as time evolves, the entanglement entropy of the subsystem A will generally start to increase, or decrease or stay the same. Explain why.

(4) 6b. The Feynman picture: Explain, briefly, under which circumstances are the rules of probabilities for alternatives and conditionals replaced by the same rules for probability amplitudes, and what are these rules? Sketch what this means for the double slit experiment.

(5) 6c. Heisenberg cut for measurements. Consider the measurement of an observable $\hat{f}$ of a quantum system. To describe the measurement procedure, we enlarge the Heisenberg cut so that the measurement instrument is included. Assume that the interaction Hamiltonian between our system and the measurement instrument is given by $\hat{H} = A \otimes B$, where A and B are self-adjoint operators that act on the Hilbert spaces of our system and of the measurement instrument respectively. Also, assume that the measurement is so brief that the free Hamiltonians of the system and the measurement instrument can be neglected, i.e., that only the interaction Hamiltonian above matters. Then, what is the observable, $\hat{f}$, that is being measured on the system? Explain your answer. [Hint: consider which states should not collapse during the measurement]