Faculty of Mathematics

University of Waterloo

AMATH473/673, PHYS454 Advanced Quantum Mechanics

Midterm Exam - Fall Term 2020

Time: 12:00pm-1:00pm + 40min for upload to Crowdmark **Date:** November 11, 2020.

Important:

(A) The solutions need to be handwritten on paper. Each page needs to be uploaded to Crowdmark twice, once plain (same as for homework), and once as a selfie with your Watcard. Example pictures are linked to on the course's home page, where the midterm is described.

(B) You are allowed to use the lecture notes for this exam. You are not allowed to communicate with others during the exam. Except, you can communicate with the instructor via email if a question is unclear.

(C) If a problem asks to give nontrivial examples then these must be examples that you yourself come up with. This means that your examples cannot be examples from textbooks and exercises or any other source that is not you. There is no need to make the examples extra complicated. But the examples must be nontrivial (e.g., not just zeros).

(D) I am very sorry to have to say this, and of course this concern doesn't apply to most of you, but plagiarism is a very serious matter because exams absolutely have to be fair for all. Therefore, I have to tell you that when a case of plagiarism is found, then I really have no choice. As a prof, I am obliged to report each and every case to the Associate Dean. Folks, please spare me having to do that! In cases where plagiarism is only suspected but not clear, the involved students may have to do an oral exam online.

(E) Your answers must be stated in a clear and logical form in order to receive full marks.

1. Explain in your own words what is the exact meaning of Δt in the time-energy uncertainty relation $\Delta t \Delta E \geq \hbar/2$.

2. a. Give an example of a 2×2 matrix that is self-adjoint.

   b. Give an example of a 2×2 matrix that is unitary.

   c. Assume that $\hat{f}$ is a self-adjoint operator and that $\hat{U}$ is a unitary operator (in an infinite-dimensional Hilbert space). Show that the operator $\hat{g} := \hat{U}^\dagger \hat{f} \hat{U}$ possesses the same eigenvalues as $\hat{f}$.

3. In the Heisenberg picture, the equations of motion obeyed by the position and momentum operators $\hat{x}(t), \hat{p}(t)$ are the same as those obeyed by the classical positions and momenta $x(t)$ and $p(t)$. However, these equations of motion are not necessarily of the same form as those obeyed by the quantum expectation values $\bar{x}(t), \bar{p}(t)$.

   a. Give an example of a Hamiltonian where they are of the same form and give an example of a Hamiltonian where they are not. (No need to prove anything to answer this question, it suffices to state examples.)

   b. What is the mathematical reason why the quantum expectation values $\bar{x}(t), \bar{p}(t)$ generally do not obey the same equations of motion as the classical variables?

   c. What is the physical intuition for why the quantum expectation values $\bar{x}(t), \bar{p}(t)$ generally do not obey the same equations of motion as the classical variables?

4. A self-adjoint operator $\hat{P}$ is called a projection operator if it obeys $\hat{P}^2 = \hat{P}$.

   a. Give an example of a nonzero projection operator.

   b. Determine which eigenvalues projection operators can possess.

   c. Assume that $\hat{f}_1$ and $\hat{f}_2$ are projectors. Under which condition(s) is $\hat{Q}_1 := \hat{f}_1 \hat{f}_2$ a projector?

   d. Assume that $\hat{g}_1$ and $\hat{g}_2$ are commuting projectors. Under which condition(s) is $\hat{Q}_2 := \hat{g}_1 + \hat{g}_2$ a projector?

5. So-called "sum rules" are a useful tool, for example, in the calculation of phenomena that involve the strong force which binds quarks together. Assume a Hamiltonian is of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ with a purely discrete and non-degenerate spectrum and with orthonormal eigenbasis $\{|E_n\rangle\}$. Prove the sum rule:
$$\sum_n (E_n - E_r)\ |\langle E_r|\hat{x}|E_n\rangle|^2 = \frac{\hbar^2}{2m}$$
   Hint: Evaluate $\langle E_r|[[\hat{x}, \hat{H}], \hat{x}]|E_r\rangle$ in two ways: One way is to write out the double commutator and to suitably insert a resolution of the identity in terms of the energy eigenbasis. The other way is to use the explicit expression that is given for $\hat{H}$.