Faculty of Mathematics
University of Waterloo
AMATH 231 - Calculus 4
Final Exam - Winter Term 2022

Time: 12:30pm-1:50pm. Plus: 40min for uploading, i.e., final deadline: 2:30pm.

Date: April 8, 2022.

Important:

(A) The solutions need to be handwritten on paper. Each page needs to be uploaded to Crowdmark twice, once plain (same as for homework), and once as a selfie with your Watcard. Example pictures are linked to on the course's home page, where the midterm is described.

(B) You are allowed to use textbooks and to Google things for this exam. You are not allowed to communicate with others during the exam. Except, you can communicate with the instructor via email if a question is unclear.

(C) In some of the following problems you are asked to give nontrivial examples. These examples must be examples that you yourself come up with. This means, that your examples cannot be examples from textbooks and exercises or any other source that is not you. There is no need to make the examples very complicated. But the examples must be nontrivial enough so that it's unlikely that anybody else chooses that example. Points can be deducted for making the examples too simple.

(D) I am very sorry to have to say this, and of course this concern doesn't apply to most of you, but plagiarism is a very serious matter because exams absolutely have to be fair for all. Therefore, I have to tell you that when a case of plagiarism is found, then I really have no choice. As a prof, I am obliged to report each and every case to the Associate Dean. Folks, please spare me having to do that! In cases where plagiarism is only suspected but not clear, the involved students may have to do an oral exam online.

(E) Your answers must be stated in a clear and logical form in order to receive full marks. For clarity, you may add a few extra words here and there to indicate what you do.

[9] 1. a) Consider the curve C given by

$$\mathbf{g}(t) = (x(t), y(t)) = (a\cos t + at\sin t, a\sin t - at\cos t), \quad -1 \leq t \leq 1$$

where a is a constant. Choose a nontrivial value for the constant a. Then calculate the length of the curve C.

b) For the same value of a, find the field lines of the vector field $\mathbf{F} = (1, 2ax)$, and illustrate them with a sketch.

c) Choose a nontrivial differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. Then calculate $u'(t)$ for the function defined through $u(t) := f(\mathbf{g}(t))$. Here, $\mathbf{g}$ is the vector-valued function from part a) above, with your choice for the constant a.

[6] 2. a) Give conditions that will guarantee that a C^1 vector field $\mathbf{F}$ is a gradient field on a subset $U \subset \mathbb{R}^3$. In your own words, explain all terminology.

b) Calculate the work done by the force

$$\mathbf{F} = 2e^{y-z}(1, x, -x)$$

acting on a particle moving along the curve C given by

$$\mathbf{g}(t) = (x(t), y(t)) = (2 - t, \cos t, \sin t),\ 0 \le t \le 3\pi.$$

[5] 3. The integral form of the law of conservation of mass in a fluid flow is

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho \, dV = -\iint\limits_{\partial\Omega} \rho \, \mathbf{v}\cdot\mathbf{n} \, dS,$$

where Ω is an arbitrary domain in $\mathbb{R}$ with piecewise smooth boundary $\partial\Omega$. The mass density ρ and fluid velocity $\mathbf{v}$ are assumed to be C^1 functions. Show that the ρ and $\mathbf{v}$ satisfy this partial differential equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0.$$

[9] 4. a) Choose a nontrivial value for the constant r. Calculate the Fourier sine series of the function

$$f(x) = r(\pi - x), \quad \text{defined on} \quad 0 < x < \pi.$$

b) Sketch the graph of the function f_p that is the sum of the series derived in a), for all $x \in \mathbb{R}$.

c) Which type of convergence does the Fourier series of f have, and why? And what value does the Fourier series f_p take at its points of discontinuity?

d) i) By using this Fourier series

$$(\pi^2 - 3t^2) = 12\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nt}{n^2}, \quad -\pi < t < \pi,$$

sum this series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n^3}.$$

ii) Sum this series:

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

[9] 5. a) Let $e_n(t) = e^{inw_0t}$, $w_0 = \frac{2\pi}{\tau}$. Evaluate $\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e_n(t)e_m(t)^* dt$, for all integers n, m. Here, * denotes complex conjugation.

b) The complex form of the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{inw_0t}.$$

Use a) to derive the expression for the Fourier coefficients c_n.

c) Use Parseval's formula

$$\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

and the Fourier series in 4 d) to sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

[8] 6. Let the Fourier transform of a function f be denoted by

$$\mathcal{F}(f(t)) = F(w)$$

a) Show that $\mathcal{F}(tf(t)) = i\frac{d}{d\omega}F(w)$.

b) Choose a random and very large integer n and calculate $\mathcal{F}(t^n f(t))$ in terms of derivatives of $F(\omega)$.

c) Derive an expression for $\mathcal{F}(\frac{d}{dt}f(t))$ in terms of $F(\omega)$.

[9] 7. a) The flux of a constant vector field through any orientable closed surface in $\mathbb{R}^3$ is zero. Explain why.

b) Assume that a vector field $\mathbf{F}$ in $\mathbb{R}^3$ is C^1 and satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere, except on the z-axis. Further assume that there exists a circle C such that $\oint_c \mathbf{F} \cdot d\mathbf{x} \neq 0$. Explain whether or not this implies that the z-axis goes through C.