Faculty of Mathematics

University of Waterloo

AMATH 231 - Calculus 4

Midterm Exam - Winter Term 2022

Thursday, February 10, 2022

Exam time and upload time: 10:00am-11:00pm exam time + 11:00-11:30am for uploading

Instructions:

(1) **During the exam, i.e., 10:00-11:00am,** your zoom video must be on, showing you, without filters or artificial background. The exam will be recorded. The solutions need to be handwritten on paper. The zoom video feed must show your face and arms but not the paper that you are writing on. You can use your mouse to scroll through the midterm, lecture notes etc. But no typing.

(2) You are allowed to use textbooks for this exam. You are not allowed to communicate with others during the exam. Except, you can communicate with the instructor via email if a question is unclear.

(3) **In the exam questions** you are asked to give nontrivial examples. These examples must be examples that you yourself come up with. This means that your examples cannot be examples from textbooks and exercises or any other source that is not you. There is no need to make the examples extra complicated. But the examples must be nontrivial enough so that nobody else has the same or almost same example. Points can also be deducted for making the examples too simple.

(4) I am very sorry to have to say this, and of course this concern doesn't apply to most of you, but cheating is a very serious matter because exams absolutely have to be fair for all. Therefore, I have to tell you that when a case of cheating is found, then I really have no choice. As a prof, I am obliged to report each and every case to the Associate Dean. Folks, please spare me having to do that! In cases where cheating is only suspected but not clear, the involved students will do an oral exam online.

(5) Your answers must be stated in a clear and logical form in order to receive full marks. Add enough words here and there to make it clear to the marker what you do.

(6) **During the upload period, i.e., 11:00-11:30am,** keep your webcam on. Each page needs to be uploaded to Crowdmark twice, once plain (as for homework), and once as a selfie with your Watcard. Example pictures of both types are linked to on the course's home page, where the link to the midterm is. You can risk using some of the upload time (11-11:30am) for calculations but the upload time cannot be extended.

[8] 1. a) Give an example of a vector-valued function that describes a nontrivial vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, i.e., $F : \ (x, y, z) \to F(x, y, z) \in \mathbb{R}^3$ (no sketching).

[8] b) Give an example of a nontrivial curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, i.e., $\gamma : t \to (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ (no sketching).

[8] c) Check by explicit calculation, while explaining what you do, (but again no sketching) whether or not the curve described by your function γ is a field line of your vector field F. (It is OK if it is not.)

2. In the xy plane, we define C to be the curve described by the path $\gamma : [0, \pi] \to \mathbb{R}^2$ given by $\gamma(t) = (e^t, \cos(2t)$.

[8] a) Give an example of a nontrivial scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ and then evaluate the line integral

$$\int_C f \; ds$$

so far that it is reduced to an ordinary integral of one variable, with no more vector operation left to be carried out. There is no need to evaluate the integral all the way to a number.

[8] b) Give an example of a nontrivial vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$. and then evaluate the integral

$$\int_C F \cdot dx$$

so far that it is reduced to an ordinary integral over one variable with no vector operations left to be carried out. There is no need to evaluate the integral all the way to a number.

[8] 3. a) Explain how to perform a test that one can do on a vector field in a simply connected region D in the plane to check if the vector field is conservative in D. (This question does not ask for the definition of what is a conservative vector field.)

[9] b) Give a nontrivial example of a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ and do the calculation that tests if your vector field is conservative in $\mathbb{R}^2$. (It is OK if it is not)

[9] 4. a) Give an example of a nontrivial and nonlinear function $g : \mathbb{R}^2 \to \mathbb{R}^3$ which has the property that $g(0,0) = (0,0,0)$. Your function g, therefore, describes a surface in $\mathbb{R}^3$ which intersects the origin, $p = (0,0,0)$, of the coordinate system.

[8] b) At the origin, $p = (0,0,0)$, of the coordinate system, calculate two tangent vectors to your surface.

[8] c) Calculate a normal vector to your surface at p.

5. Consider the vector field $\vec{F}(x,y) = (-x + 14y^2)\,\hat{\imath} + 28xy\,\hat{\jmath}$.

[9] a) Find a scalar field $\Phi(x,y)$ (i.e., a potential) of which $\vec{F}$ is the gradient field.

[9] b) Evaluate the line integral $\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$ along the curve parameterized by

$$\vec{\gamma}(t) = a(2\cos t - \cos 2t)\,\hat{\imath} + a(2\sin t - \sin 2t)\,\hat{\jmath}$$

with $t \in [0, 2\pi]$, $(a > 0)$.