arXiv:gr-qc/0403112v1 30 Mar 2004

SHORT-DISTANCE CUTOFFS IN CURVED SPACE*

A. KEMPF[†]

Department of Applied Mathematics, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
akempf@uwaterloo.ca

It is shown that space-time may possess the differentiability properties of manifolds as well as the ultraviolet finiteness properties of lattices. Namely, if a field's amplitudes are given on any sufficiently dense set of discrete points this could already determine the field's amplitudes at all other points of the manifold. The criterion for when samples are sufficiently densely spaced could be that they are apart on average not more than at a Planck distance. The underlying mathematics is that of classes of functions that can be reconstructed completely from discrete samples. The discipline is called sampling theory and is at the heart of information theory. Sampling theory establishes the link between continuous and discrete forms of information and is used in ubiquitous applications from scientific data taking to digital audio.

1. Introduction

General relativity and quantum theory, when considered together, are indicating that the notion of distance loses operational meaning at the Planck scale of about $10^{-35}m$ (assuming 3+1 dimensions). Namely, if one tries to resolve a spatial structure with an uncertainty of less than a Planck length, then the corresponding momentum uncertainty should randomly curve and thereby significantly disturb the very region in space that is meant to be resolved. To obtain a unified theory of general relativity and quantum theory is difficult, however, not least because the two theories are formulated in the very different languages of differential geometry and functional analysis.

One of the problems in the effort to find a unifying theory of quantum gravity is, therefore, to develop a mathematical framework which combines differential geometry and functional analysis such as to give a precise description of a notion of a shortest distance in nature. Candidate theories may become testable when introduced to inflationary cosmology and compared to CMB measurements, see [1].

There has been much debate about whether the unifying theory will describe space-time as being discrete or continuous. It is tempting, also, to speculate that a quantum gravity theory such as M theory, see [2], or a foam theory, see [3], once

---

2

fully understood, might reveal the structure of space-time as being in some sense in between discrete and continuous, possibly such as to combine the the differentiability of manifolds with the ultraviolet finiteness of lattices. This third possibility seems to be ruled out, however: as Gödel and Cohen proved, no set can be explicitly constructed whose cardinality would be in between discrete and continuous, see for example [4]. Nevertheless, there still is at least one possibility by which space-time might have a mathematical description which combines the differentiability of manifolds with the ultraviolet finiteness of lattices:

2. Fields with a finite density of degrees of freedom

Let us recall that physical theories are formulated not directly in terms of points in space or in space-time but rather in terms of the functions in space or in space-time. This suggests a whole new class of mathematical models for a finite minimum length. Namely, fields in space-time could be functions over a differentiable manifold as usual, while the class of physical fields is such that if a field is sampled only at discrete points then its amplitudes can already be reconstructed at *all* points in the manifold - if the sampling points are spaced densely enough. The maximum average sample spacing which allows one to reconstruct the continuous field from discrete samples could be on the order of the Planck scale, see [5].

Since any one of all sufficiently tightly spaced lattices would allow reconstruction, no particular lattice would be preferred. It is because no particular lattice is singled out that the symmetry properties of the manifold can be preserved.

The physical theory could be written, equivalently, either as living on a differentiable manifold, thereby displaying e.g. external symmetries, or as living on any one of the sampling lattices of sufficiently small average spacing, thereby displaying its ultraviolet finiteness. Physical fields, while being continuous or even differentiable, would possess only a finite density of degrees of freedom.

The mathematics of classes of functions which can be reconstructed from discrete samples is well-known, namely as *sampling theory*, in the information theory community, where it plays a central role in the theory of sources and channels of continuous information as developed by Shannon, see [6].

3. Sampling theory

The simplest example in sampling theory is the Shannon sampling theorem: Choose a frequency ω_{max}. Consider the class $B_{\omega_{max}}$ of continuous functions f whose frequency content is limited to the interval $(-\omega_{max}, \omega_{max})$, i.e. for which: $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} = 0$ whenever $|\omega| \geq \omega_{max}$. If the amplitudes $f(x_n)$ of such a function are known at equidistantly spaced discrete values $\{x_n\}$ whose spacing is π/ω_{max} or smaller, then the function's amplitudes $f(x)$ can be reconstructed

for all x. Shannon's reconstruction formula reads:

$$f(x) \; = \; \sum_{n=-\infty}^{\infty} f(x_n) \, \frac{\sin[(x-x_n)\omega_{max}]}{(x-x_n)\omega_{max}} \tag{1}$$

The theorem is in ubiquitous use in digital audio and video as well as in scientific data taking. Sampling theory, see [7], studies generalizations of the theorem for various different classes of functions, for non-equidistant sampling, for multivariable functions and it investigates the effect of noise, which could be quantum fluctuations in our case. As was shown in [5], generalized sampling theorems automatically arise from stringy uncertainty relations, namely whenever there is a finite minimum position uncertainty Δx_{min}, as e.g. in uncertainty relations of the type: $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...)$, see [8]. A few technical remarks: the underlying mathematics is that of symmetric non self-adjoint operators. Through a theorem of Naimark, unsharp variables of POVM type arise as special cases.

Due to the origin of sampling theory in engineering, sampling theory for generic (pseudo-) Riemannian manifolds has been undeveloped so far. Here, I suggest a general approach to this problem.

4. Information Theory on Curved Space

Let us consider as a natural (because covariant) analogue of the bandwidth restriction of the Shannon sampling theorem in curved space the presence of a cutoff on the spectrum of the Laplace operator $-\Delta$ on a Riemannian manifold $\mathbf{M}$. (On a pseudo-Riemannian or a spin manifold, we can consider the d'Alembert or the Dirac operator, to obtain a covariant form of sampling theory. This is pursued in [9].)

We start with the usual Hilbert space $\mathbf{H}$ of square integrable scalar functions over the manifold, and we consider the dense domain $\mathbf{D} \subset \mathbf{H}$ on which the Laplacian is essentially self-adjoint. Using physicists' sloppy but convenient terminology we will speak of all points of the spectrum as eigenvalues, λ, with corresponding "eigenvectors" $|\lambda)$. Since we are mostly interested in the case of noncompact manifolds, whose spectrum will not be discrete, some more care will be needed, of course. For Hilbert space vectors we use the notation $|\psi)$, in analogy to Dirac's bra-ket notation, only with round brackets.

Let us define P as the projector onto the subspace spanned by the eigenspaces of the Laplacian with eigenvalues smaller than some fixed maximum value λ_{max}. (For the d'Alembertian and for the Dirac operator, let λ_{max} bound the absolute values of the eigenvalues.)

We consider now the possibility that in nature all physical fields are contained within the subspace $\mathbf{D}_s = P.\mathbf{D}$, where λ_{max} might be on the order of $1/l_{Planck}^2$. In fact, through this spectral cutoff, each function in $\mathbf{D}_s$ acquires the sampling property: if its amplitude is known on a sufficiently dense set of points of the manifold, then it can be reconstructed everywhere. Thus, through such a spectral cutoff a sampling theorem for physical fields arises naturally. To see this, assume for

4

simplicity that one chart covers the N-dimensional manifold. Consider the coordinates $\hat{x}_j$, for $j = 1, ..., N$ as operators that map scalar functions to scalar functions: $\hat{x}_j : \phi(x) \to x_j \phi(x)$. On their domain within the original Hilbert space $\mathbf{H}$, these operators are essentially self-adjoint, with an "Hilbert basis" of non-normalizable joint eigenvectors $\{|x)\}$. We can write scalar functions as $\phi(x) = (x|\phi)$, i.e. scalar functions are the coefficients of the abstract Hilbert space vector $|\phi) \in \mathbf{H}$ in the basis of the vectors $\{|x)\}$. The continuum normalization of the $|x)$ is with respect to the measure provided by the metric. On the domain of physical fields, $\mathbf{D}_s$, the multiplication operators $\hat{x}_j$ are merely symmetric but not self-adjoint. The projections $P|x)$ of the eigenvectors $|x)$ onto the physical subspace $\mathbf{D}_s$ are in general no longer orthogonal. Correspondingly, the uncertainty relations of the first quantization must be modified, see [10].

Consider now a physical field, i.e. a vector $|\phi) \in \mathbf{D}_s$, which reads as a function: $\phi(x) = (x|\phi)$. Assume that only at the discrete points $\{x_n\}$ the field's amplitudes $\phi(x_n) = \langle x_n|\phi\rangle$ are known. Then, if the discrete sampling points $\{x_n\}$ are sufficiently dense, they fully determine the Hilbert space vector $|\phi\rangle$, and therefore $\phi(x)$ everywhere. To be precise, we assume the amplitudes

$$\phi(x_n) = (x_n|\phi) = \sum\!\!\!\!\!\!\int_{|\lambda| < \lambda_{max}} (x_n|\lambda)(\lambda|\phi) \tag{2}$$

to be known. We use the sum and integral notation because $\{\lambda\}$ may be discrete and or continuous (the manifold $\mathbf{M}$ may or may not be compact). Define $K_{n\lambda} = (x_n|\lambda)$. The set of sampling points $\{x_n\}$ is dense enough for reconstruction iff K is invertible, because then: $(\lambda|\phi) = \sum_n K_{\lambda,n}^{-1}\phi(x_n)$. We obtain the reconstruction formula:

$$\phi(x) = \sum_n \left(\sum\!\!\!\!\!\!\int_{|\lambda| < \lambda_{max}} (x|\lambda) K_{\lambda n}^{-1} \right) \phi(x_n) \tag{3}$$

It should be interesting to investigate the stability of the reconstruction of samples along the lines of the work by Landau [11]. The reconstruction stability is usually of importance due to noise. Here the question arises to which extent quantum fluctuations may play the role of noise. Also, following Shannon and Landau, it appears natural to define the density of degrees of freedom through the number of dimensions of the space of functions in $\mathbf{D}_s$ with essential support in a given volume. Clearly, we recover conventional Shannon sampling as a special case. The Shannon case has been applied to inflationary cosmology in [1] and it should be very interesting to apply to cosmology also the more general approach presented here. We note that higher than second powers of the fields (second powers occur as scalar products in the Hilbert space of fields) are now nontrivial to enter into a quantum field theoretical action: To this end, the multiple product of fields needs to be defined such as to yield a result within the cut-off Hilbert space. In this context it should be interesting also to reconsider the mechanism of Sakharov's induced gravity, see [12].

Our approach to sampling on curved space significantly simplifies in the case of compact manifolds, where the spectrum of the Laplacian is discrete and the cut off Hilbert space $\mathbf{H}_s$ is finite dimensional. Intuitively, it is clear that knowledge of a function at as many points as is the dimension of the cutoff Hilbert space generically allows one to reconstruct the function everywhere. Results of spectral geometry, see e.g. [13,14,15,16] are likely to be useful also for the approach described here.

5. Vacuum energy

Vacuum energy is of great current cosmological interest and it will be interesting to use the covariant cutoff tools here to investigate this problem. Regarding the vacuum energy, I would like to close with a speculative thought: In the Casimir effect, the system's energy can be lowered not only by moving the plates closer but instead also by making plates which are at a fixed distance into better conductors, as this increases the strength and frequency range of their mode-expelling property. It should be interesting to investigate if the copper oxide "plates" in High-Tc superconductors can be viewed as being driven towards superconductivity by the reduction in energy through the Casimir effect.

References

1. A. Kempf, 2001, Phys.Rev.**D63** 083514, astro-ph/0009209, A. Kempf, J. C. Niemeyer, 2001, Phys.Rev.**D64** 103501, astro-ph/0103225, R. Easther, B. R. Greene, W. H. Kinney, G. Shiu, 2001, Phys.Rev.**D64** 103502, hep-th/0104102, R. Easther, B. R. Greene, W. H. Kinney, G. Shiu, 2002, Phys.Rev.**D66** 023518, hep-th/0204129
2. J. Polchinski, hep-th/0209105
3. F. Markopoulou, gr-qc/0203036, D. Oriti, H. Pfeiffer, gr-qc/0207041, L. Smolin, hep-th/0209079
4. A. Kempf, in Proceedings 18th IAP Colloquium on the Nature of Dark Energy, Paris, France, 1-5 Jul 2002. e-Print Archive: gr-qc/0210077
5. A. Kempf, 2000, Phys.Rev.Lett. **85**, 2873, hep-th/9905114, A. Kempf, 1997, Europhys.Lett. **40** 257, hep-th/9706213
6. C. E. Shannon, W. Weaver, 1963, *The Mathematical Theory of Communication*, Univ. of Illinois Press.
7. J.J. Benedetto, P.J.S.G. Ferreira, 2001, *Modern Sampling Theory*, Birkaeuser
8. E. Witten, 1996, Phys. Today **49** 24, A. Kempf, 1994, J. Math. Phys. **35**, 4483
9. gr-qc/0310035
10. A. Kempf, hep-th/9311147, J.Math.Phys. **35** 4483 (1994), A. Kempf, G. Mangano, R. B. Mann, hep-th/9412167, Phys.Rev.**D52** 1108 (1995)
11. H.J. Landau, Proc. of the IEEE, **10**, 1701 (1967)
12. A. D. Sakharov, reprinted in Gen. Rel. Grav. **32**, 365 (2000)
13. B. Davies, Y. Safarov, *Spectral Theory and Geometry*, Cambridge Univ. Press (1999)
14. P. B. Gilkey, J. V. Leahy, J. Park, *Spinors, Spectral Geometry, and Riemannian Submersions*, Electronic Library of Mathematics, http://www.emis.de/ELibEMS.html (1998)
15. G. Esposito, *Dirac Operators and Spectral Geometry*, Cambridge Univ. Press (1998)
16. M. Puta, T. M. Rassias, M. Craioveanu, *Old and New Aspects in Spectral Geometry*, Kluwer (2001)