Note Title

QFT for Cosmology, Achim Kempf, Winter 2014, Lecture 8

The Unruh effect

(W.G. Unruh, 1976)

(Can be interpreted as showing that the very existence or non-existence of particles is sometimes observer dependent.)

The Unruh effect is the observation, by accelerated observers, of particles, even when the field is in the vacuum state in Minkowski space, i.e., even if inertial observers don't see particles.

Intuition 1:

- A monochromatic wave in an inertial frame is not monochromatic for an accelerated observer.
- Thus, the accelerated observer's modes are coupled oscillators: he sees wavelengths change.
- These oscillator's ground state is now different.

→ Calculation strategy 1:

- Use accelerated observers' mode decomposition.
- Relate it to inertial observer's mode decomposition.
- Choose vacuum for the inertial observer
- Calculate particle production for accelerated observer analogous to $|n_{in}\rangle$ to $|n_{out}\rangle$ transform for driven harmonic oscillators' evolution above.

Intuition & Strategy 2: (for a change, we will pursue this one here)

- Consider an accelerated detector of particles.
- Detector := a quantum system <u>coupled</u> to the field.
- Detection := detector's state goes from ground state to an excited state.
- The detector has an accelerated charge for the field.

$\Rightarrow$ We expect field excitation, i.e. radiation of particles.

$\Rightarrow$ Field acts on detector $\Rightarrow$ detector excitation, i.e. Unruh effect.

An accelerated particle detector:

- Consider an observer with a particle detector.

- **Definition:** Let τ be the eigentime of the accelerated observer and detector.

- **Definition:** We write the accelerated path as

$$x^\mu(\tau) = \left(x^0(\tau), \vec{x}(\tau)\right)$$

- **Note:** Here, x^0 and $\vec{x} = (x^1, x^2, x^3)$ are the observer and detector's coordinates in a cartesian coordinate system of an inertial observer.

□ Examples:

* An observer at rest has: $x^\mu(\tau) = (\tau, 0, 0, 0)$

* Case of constant velocity:

$$x^\mu(\tau) = (a\tau, \vec{b}\tau)$$

with $a^2 - \vec{b}^2 = 1$. Exercise: verify

* Case of constant acceleration in the x-direction:

$$x^0(\tau) = \alpha \sinh(\tau/\alpha)$$

$$x^1(\tau) = \alpha\,(1 + \sinh(\tau/\alpha))^{1/2}$$

$$x^2(\tau) = x^3(\tau) = 0$$

Exercise: □ verify that $\ddot{x}^\mu \ddot{x}_\mu = \text{const}$

□ show that for $\tau \ll 1$: $x(\tau) \simeq (\tau, a + b\tau^2)$ (i.e. for still small velocities)

The quantum field

- We assume that the detector can detect particles of a Klein Gordon field $\hat{\phi}$.

- We assume that, for an inertial observer the field $\hat{\phi}$ is in the Minkowski vacuum. **Recall:**

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\vec{x}}\, \hat{\phi}_k(x^0)\, d^3k \quad \text{with} \quad \hat{\phi}_k(x^0) = \frac{1}{\sqrt{2}} \left(V_k^*(x^0)\, a_{\vec{k}} + V_{-k}(x^0)\, a_{-k}^+ \right)$$

and $V_k(x^0) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0}$ with $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

- Thus:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 + i\vec{k}\vec{x}}\, a_k + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\vec{x}}\, a_{-k}^+ \right) d^3k$$

The accelerated detector

- Assume that the accelerated observer's detector is a quantum system with discrete energy levels:

$$E_0, E_1, E_2, \ldots$$

- Examples:
  - An atom
  - An oscillator such as the diatomic molecule H_2.

- The quantum system thus consists of two subsystems, with these Hamiltonians:

$$\hat{H}^{total} = \hat{H}_0^{detector} \otimes 1 + 1 \otimes \hat{H}_0^{field} + \hat{H}^{interaction}$$

- The Hilbert space factorizes:

$$\mathcal{H}^{total} = \mathcal{H}^{detector} \otimes \mathcal{H}^{field}$$

- The interaction Hamiltonian $\hat{H}^{interaction}$ consists of operators of both subsystems, for example:

$$\hat{H}^{interaction}(\tau) = \varepsilon(\tau)\,\hat{Q}(\tau)\,\hat{\phi}(x^0(\tau), \vec{x}(\tau))$$

Detector efficiency (can also be used as on/off switch)

An observable of the detector's quantum system

The field $\hat{\phi}$ at the current detector location

- Examples: $\hat{H}^{int}(\tau) = \hat{S}_3(\tau)\,\hat{B}_3(x(\tau))$

detector is a spin. field is magnetic field.

or: $\hat{H}^{int}(\tau) = -\frac{e}{mc}\,\hat{p}_i\,\hat{A}^i(x(\tau))$ (use Axial gauge: $\partial_i A^i = 0$)

Time evolution

- If we (realistically) assume that the detector efficiency $\varepsilon(\tau)$ is small, we can use perturbation theory.

- In this case, the Dirac picture of time evolution is convenient:

  * Operators evolve according to

$$\hat{H}^{free} = \hat{H}^{detector} \otimes 1 + 1 \otimes \hat{H}^{field}$$

For example:

$$\hat{Q}(\tau) = e^{i\hat{H}^{free}\tau} (\hat{Q}_0 \otimes 1)\, e^{-i\hat{H}^{free}\tau}$$

$$= e^{i\hat{H}^{detector}\tau}\, \hat{Q}_0\, e^{-i\hat{H}^{detector}\tau} \otimes 1$$

* States evolve according to $\hat{H}^{\text{int}}(\tau)$, i.e., according to the time evolution operator:

$$\hat{U}(\tau) = T\exp\left(i\int_{\tau_i}^{\tau_f} \hat{H}^{\text{interaction}}(\tau')\,d\tau'\right)$$

the time-ordering symbol

Perturbative ansatz

- For small detector efficiency $\varepsilon(\tau)$ we can expand:

$$\hat{U}(\tau) = 1 + i\int_{-\infty}^{\tau} \varepsilon(\tau')\,\hat{Q}(\tau')\,\hat{\phi}(x^0(\tau'),\vec{x}(\tau'))\,d\tau' + \mathcal{O}(\varepsilon^2)$$

- **Note**: We can set $\tau_i = -\infty$ since we can always switch $\varepsilon(\tau)$ on or off.

Initial conditions

- We assume that the quantum field $\hat{\phi}$ starts out in a state $|\alpha\rangle$ with $|\alpha\rangle$ = Minkowski vacuum, $|\alpha\rangle = |0\rangle$, or a 1-particle state $|\alpha\rangle = |1_k\rangle$.

- We assume that the detector starts out in its ground state $|E_0\rangle$.

- Thus, the combined system starts out in the state:

$$|\Psi_{in}\rangle = |E_0\rangle \otimes |\alpha\rangle$$

- Time evolution:

At time τ the total system is in the state

$$|\Psi(\tau)\rangle = \hat{U}(\tau)\,|\Psi_{in}\rangle$$

Particle creation

- **The problem:**

  What is the probability amplitude that, if we measure at time τ the detector system will be found to have detected something, i.e., to be in an excited state $|E_n\rangle$?

- To this end, calculate:

$$p(\tau) := \left(\langle E_n| \otimes \langle \Omega|\right) |\psi(\tau)\rangle$$

  for an arbitrary end state $|\Omega\rangle$ of the quantum field $\hat{\phi}$.

- **Note:** We will see that not all states $|\Omega\rangle$ yield a nonzero $p(\tau)$.

Total detection probability:

- The probability for detection eventually is:

(we may choose $\varepsilon(\tau)$ so as to make it finite)

$$p(\infty) \approx \langle E_n| \otimes \langle \Omega| \left(1 + i \int_{-\infty}^{+\infty} \varepsilon(\tau') \, \hat{Q}(\tau') \, \hat{\phi}(x(\tau')) \, d\tau' \right) |E_0\rangle \otimes |\alpha\rangle$$

Note: $\langle E_n | E_0 \rangle = 0 \implies 1^{st}$ term vanishes $\implies$

$$= i \int_{-\infty}^{+\infty} \varepsilon(\tau) \langle E_n | \hat{Q}(\tau) | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \alpha \rangle \, d\tau$$

Recall: $\hat{Q}(\tau) = e^{i H_0^{detector} \tau} \, \hat{Q}_0 \, e^{-i H_0^{detector} \tau}$

$$= i \int_{-\infty}^{+\infty} \varepsilon(\tau)\, e^{i(E_n - E_0)\tau} \langle E_n | \hat{Q}_0 | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \alpha \rangle \, d\tau$$

Recall: $\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 - i\vec{k}\vec{x}} a^+_{\vec{k}} + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\vec{x}} a_{\vec{k}} \right) d^3k$

The case $|\alpha\rangle = |0\rangle$:

* In $\hat{\phi}(x)$, only the terms $\sim a^+_k$ can contribute, because $a_k|0\rangle = 0$

* Thus, in $|\Omega\rangle$ only the one-particle components contribute:

$$|\Omega\rangle = \cancel{\Omega_0 |0\rangle} + \int \Omega_k a^+_k |0\rangle d^3k + \cancel{\iint \Omega_{kk'} a^+_k a^+_{k'} |0\rangle d^3k\, d^3k' + \ldots}$$

* Thus, let us consider a $|\Omega\rangle = a^+_{\vec{k}} |0\rangle$:

$$\Rightarrow \quad p(\infty) \simeq i\frac{\langle E_n|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\varepsilon(\tau)\,e^{i(E_n-E_0)\tau}\langle 0|a_{\tilde{k}}\int\frac{e^{i\omega_k x_0(\tau)-i\vec{k}\vec{x}(\tau)}}{\sqrt{2\omega_k}}a_k^+\,d^3k|0\rangle\,d\tau$$

leads to:

$$\langle 0|a_{\tilde{k}}a_k^+|0\rangle = \langle 0|a_k^+a_{\tilde{k}} + \delta^3(\tilde{k}-k)|0\rangle = \delta^3(\tilde{k}-k)$$

$$\Rightarrow \quad p(\infty) = \underbrace{i\frac{\langle E_n|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}}}_{\text{some constant}}\int_{-\infty}^{+\infty} e^{i(E_n-E_0)\tau}\,e^{i(\omega_k x^0(\tau)-\tilde{k}\vec{x}(\tau))}\,\varepsilon(\tau)\,d\tau$$

Special case: $|\alpha\rangle = |0\rangle$ and detector inertial:

* Choose the detector's rest frame: $x^\mu(\tau) = (\tau, 0, 0, 0)$

* Thus:

$$p(\infty) = i\frac{\langle E_n|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} e^{i(E_n-E_0)\tau}\, e^{i(\omega_{\tilde{k}} x^0(\tau) - \tilde{k}\vec{x}(\tau))}\, \varepsilon(\tau)\, d\tau$$

assume $\varepsilon(\tau)=1$, i.e., "always on".

$$= i\frac{\langle E_n|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} e^{i(E_n-E_0)\tau}\, e^{i\,\omega_{\tilde{k}}\tau}\, d\tau$$

$$= i\frac{\langle E_n|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}}(2\pi)^{1/2}\,\delta\Big(\underbrace{\overbrace{E_n - E_0}^{>0} + \overbrace{\omega_{\tilde{k}}}^{\sqrt{k^2+m^2}>0}}_{\text{this cannot be } 0}\Big)$$

$$= 0$$

⇒ No excitation of the detector, as expected.

Special case: $|\alpha\rangle = |0\rangle$ and detector non-inertial:

- The probability amplitude for the detector to become excited will depend on the excitation energy:

$$E_{ex} := E_n - E_0$$

- Namely:

$$p(\infty) = \underbrace{i\frac{\langle E_n|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}}}_{\text{a constant}} \int_{-\infty}^{+\infty} \underbrace{e^{i(E_n-E_0)\tau}}_{\substack{\text{Fourier factor} \\ \text{i.e. } \tau \text{ and } E_{ex} \\ \text{are a Fourier pair} \\ \text{(if neglecting the "constant")}}} \underbrace{e^{i(\omega_k x^0(\tau) - \tilde{k}\vec{x}(\tau))}}_{\substack{\text{function that is being} \\ \text{Fourier transformed}}} \varepsilon(\tau)\, d\tau$$

□ **Clearly:**

For generic, accelerated detectors the function

$$f(\tau) := e^{i(\omega_k x^0(\tau) - \tilde{k}\vec{x}(\tau))} \, \varepsilon(\tau)$$

possesses a Fourier transform

$$F(E_x) = \int_{-\infty}^{+\infty} e^{iE_x \tau} f(\tau) \, d\tau \,, \qquad E_x = E_n - E_0$$

which is generally nonzero for positive E_x.

$\Rightarrow$ $p(\infty) \sim F(E_x) \neq 0$ $\Rightarrow$ detector does get excited.

"proportional to" (European notation)

(while also the field gets excited)

Example: The constantly accelerated detector.

* One finds that the prob. of excitation is identical to the case in which the detector is in a heat bath of temperature $T \sim \alpha$ where α is the acceleration.

* For details, see e.g. text by Birrel & Davies.

Remark:

* Note that both the detector and the quantum field become excited. Is energy conservation violated?

* One can show that the energy stems from the accelerating agent:

* It's the case of an system with charge in time-dependent interation with the field: An antenna where field & system get excited.

E.g.: Think of a regular antenna. If the accelerated e^- were excitable little systems, they would get excited.

Special case: $|\alpha\rangle = |1_k\rangle$:

Recall:

Prob. amplitude for detector to get excited

$$P = i\int_{-\infty}^{+\infty} \varepsilon(\tau)\, e^{i(E_n - E_0)\tau} \langle E_n|\hat{Q}_0|E_0\rangle \langle \Omega|\hat{\phi}(x(\tau))|\alpha\rangle \, d\tau$$

Recall: $$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 - i\vec{k}\vec{x}} a^{+}_{\vec{k}} + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\vec{x}} a_{\vec{k}} \right) d^3k$$

$\Rightarrow$ For $|\alpha\rangle = |1_k\rangle = a^{+}_{k}|0\rangle$, we can have:

a.) $|\Omega\rangle = |2_k\rangle$, or $|1_k, 1_{\tilde{k}}\rangle$ Would mean detector excites the field further

i.e., not only "detects" a particle.

b.) $|\Omega\rangle = |0\rangle$: Means detector absorbs a particle.