

Realistic cosmic inflation

1. How can a period of near-exponential expansion be caused?

□ Recall the full action:

We neglect such terms by Occam's razor: there is no evidence for their existence as yet.

$$S = - \frac{1}{16\pi G} \int \left[2\Lambda + R(x) + \cancel{\mathcal{O}(R\phi)} + \cancel{\mathcal{O}(R^2)} + \dots \right] \sqrt{|g|} d^4x$$

↑ cosm. constant

Note: ϕ is now called the "Inflaton" field.

$$+ \int \left[\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right] \sqrt{|g|} d^4x$$

+ ~~Other fields~~

← We neglect this term because the contribution of the inflaton field ϕ and of $g_{\mu\nu}$ are assumed to have been dominant in the very early universe.

Example choice of V : $V(\phi) = m\phi^2 + \lambda\phi^4$

□ Equations of motion:

* $\frac{\delta S}{\delta \phi(x)} = 0$ yields the K.G. eqn.:

$$\frac{\partial}{\partial x^\mu} \left(g^{\mu\nu}(x) \phi_{,\nu}(x) \sqrt{|g(x)|} \right) + \frac{\partial V}{\partial \phi}(x) \sqrt{|g(x)|} = 0 \quad (KG)$$

* $\frac{\delta S}{\delta g_{\mu\nu}(x)} = 0$ yields the Einstein eqn.:

$$R_{\mu\nu}(x) - g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = -8\pi G T_{\mu\nu}(x) \quad (E)$$

where the energy-momentum tensor (for ϕ only) reads:

$$T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left(g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi) \right) + \cancel{T_{\mu\nu}^{(other fields)}}$$

We'll assume this is small compared to the contribution of ϕ , during the very early universe.

The important special case of homogeneity & isotropy

Eqs. (K.G) and (E) are a set of coupled nonlinear partial differential equations which are even classically very hard.

→ As a lowest order approximation we assume perfect homogeneity & isotropy:

$$\phi(x,t) = \phi(t)$$

$$g_{\mu\nu}(x,t) = g_{\mu\nu}(t)$$

Note:

This may also be viewed as considering only the $k=0$ modes, neglecting all other modes.

Thus, the eqns of motion simplify:

$$\left(\cdot = \frac{\partial}{\partial t}\right) \quad \ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \frac{dV}{d\phi} = 0 \quad (\text{K.G. eqn.})$$

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G T_0^0 + \Lambda \quad \left(\begin{array}{l} \text{the } 0,0 \text{ component of} \\ \text{the Einstein equation} \end{array}\right)$$

$$-2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi G T_i^i - \Lambda \quad \left(\begin{array}{l} \text{the } i,i \text{ components of} \\ \text{the Einstein equation} \end{array}\right)$$

↖ no sum

$$\text{Here: } T_0^0 = \mathcal{E}(t) = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (\text{the energy density } \mathcal{E} \text{ of } \phi)$$

$$T_i^i = p(t) = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (\text{the pressure } p \text{ of } \phi)$$

Given any initial conditions and given any $V(\phi)$ one can now solve for $a(t)$, $\phi(t)$, at least numerically!

Notice:

- The cosmological constant Λ contributes effectively a positive energy density ρ_Λ and effectively a negative pressure p_Λ .
 - Vice versa, whenever $V(\phi) \gg \dot{\phi}^2/2$ then $V(\phi)$ temporarily plays the same rôle as Λ .
 - How close we are to $V(\phi) \gg \dot{\phi}^2/2$ is described by the "Equation of state parameter":
$$w(t) := \frac{p(t)}{\rho(t)} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} \quad -1 < w < 1$$
- $\Rightarrow \int w \approx -1$ then $V(\phi)$ acts like a cosm. constant.

First attempt to get exponential expansion:

Assume that Λ dominates over $T_{\mu\nu}$ of all fields in nature.

Then, the 0,0 component of Einstein's equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} T_0^0 + \frac{1}{3} \Lambda \quad \text{becomes} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3} \Lambda$$

whose solution has the desired behavior:

$$a(t) = a_0 e^{Ht} \quad \text{with} \quad H = \sqrt{\Lambda/3} !$$

Problems: □ Λ is too tiny! It is 122 orders of magnitude below the Planck scale. We'd need a Λ close to the Planck scale 10^{70} m^{-2} . (just a few orders of magnitude below)

Note:
 Λ manifests itself as dark energy and has been measured:
 $\Lambda \approx 10^{-52} \text{ m}^{-2}$

□ Since Λ is constant, such an inflation would never end!

→ Realistic possibility:

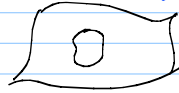
$V(\phi)$ temporarily very large

→ One of the biggest ideas of science, ever:

□ Consider a universe like ours.

Everywhere, at all times, all fields quantum fluctuate.

□ As a rare fluke, the field ϕ quantum fluctuates in a patch a few Planck lengths in size

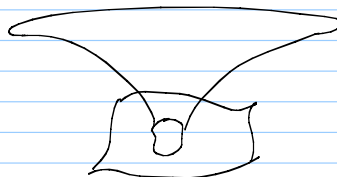


to a ϕ value that makes $V(\phi)$ close to the Planck scale.

(Assume homogeneity in that patch, so that the $\partial_i \phi$ are small)

□ In this patch, the equations above hold, with $V(\phi)$ dominant and imparting $a(t)$ like a large Λ would

⇒ Before the fluctuation can "snap back", general relativity will quasi-exponentially inflate this patch (potentially, e.g., by 10^5 orders of magnitude).



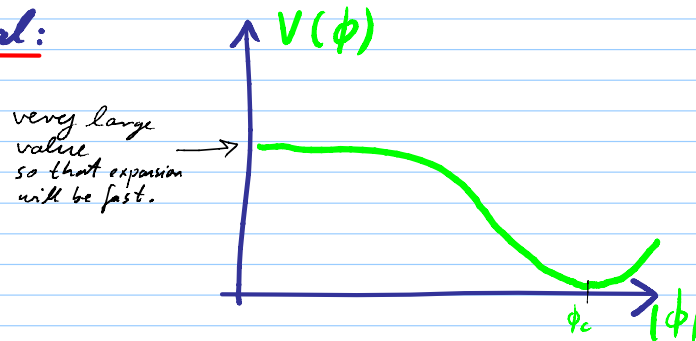
⇒ The mother universe spawns a daughter universe!

□ $V(\phi)$ in the patch starts out high but will dynamically fall eventually to low value → Inflation ends.

□ The energy in $V(\phi)$ turns into hot matter.

□ The daughter universe can spawn new universes and so on...

Example potential:



- Then, inflation starts when, in a patch, ϕ is very small, even though it is energetically expensive (a rare quantum fluctuation.)
- Then, after ϕ starts out at $\phi=0$ and large $V(\phi)$, it will slowly evolve towards ϕ_c while the universe inflates, thus flattens, and the matter dilutes.
- Once $\phi = \phi_c$ is reached, $V(\phi)=0$, and inflation has ended.

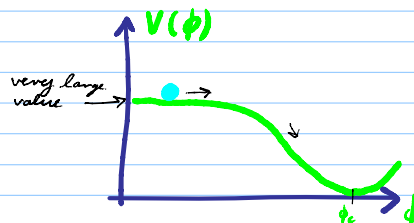
* Concretely:

The Klein Gordon equation reads:

$$\ddot{\phi} = -3 \frac{\dot{a}}{a} \dot{\phi} - \frac{dV}{d\phi}$$

↙ friction term

This is like the equation of motion of a ball rolling down a hill, with friction:



$\left(-\frac{dV}{d\phi}\right)$ acts to pull ϕ down the potential hill.

$\left(-3 \frac{\dot{a}}{a} \dot{\phi}\right)$ acts as a "friction" term.

* Definition:

If the initial value of $V(\phi)$ is very large and if the initial slope is very flat, i.e., if the ball for a period rolls slowly, with approximately constant $V(\phi)$, we call this a period of "Slow Roll Inflation".

* Observation:

During the slow roll period, we have, in particular, that $V(\phi)$ dominates over $\frac{1}{2}\dot{\phi}^2$.

$$\Rightarrow \quad w(t) = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \approx -1 \quad (\text{temporarily})$$

* But, do we also get temporary exponential inflation?

Indeed, the 0,0 component of the Einstein equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3}\Lambda = \frac{8\pi G}{3}T_0$$

is during the slow roll period:

$$\left(\frac{\dot{a}}{a}\right)^2 \approx \frac{8\pi G}{3}V(\phi)$$

whose solution during slow roll is

$$a(t) \approx a_0 e^{\pm \sqrt{\frac{8\pi G}{3}V(\phi)} t} \quad \left(\phi \text{ and } V(\phi) \text{ change slowly over time in slow roll.}\right)$$

* Thus, during slow roll, we have effectively a slowly varying Hubble parameter!

Definition:

The function $H(t) := \frac{\dot{a}(t)}{a(t)}$ is called the Hubble parameter function.

* In the case $a(t) = e^{Ht}$ we recover $H = H(t)$.

* In the case of slow roll inflation, we have

$$H(t) = \sqrt{\frac{8\pi G}{3} V(\phi(t))}$$

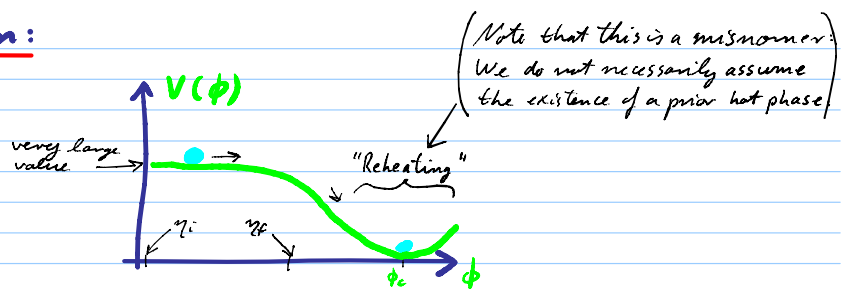
Remark:

□ As $V(t)$ decreases, also $H(t)$ decreases.

⇒ inflation predicts that $\delta\phi_L$ decreases for later and later horizon crossing modes, i.e., for smaller and smaller wavelength modes.

The WMAP satellite's CMB data show evidence for this!

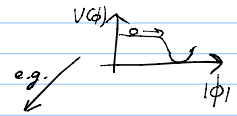
The end of inflation:



* In the period called "Re-heating", the energy of ϕ is transferred into the mode oscillators of the usual (low mass) particles, i.e., the inflaton particles decay and thereby create a high energetic, i.e. hot, plasma of literally all sorts of particles.

* From thereon, the usual big bang cosmology is assumed to have followed.

Quantum fluctuations during cosmic inflation



Strategy:

□ We assume a suitable potential $V(\phi)$ and suitable initial conditions, as discussed before.

⇒ Solutions $\phi_0(t)$ and $a_0(t)$ which exhibit slow roll inflation for a suitable finite time interval $[t_i, t_f]$, i.e., $[\eta_i, \eta_f]$.

□ We consider the case of small inhomogeneities in the inflaton field:

$$\phi(x, \eta) = \phi_0(\eta) + \psi(x, \eta) \quad \text{with} \quad |\psi(x, \eta)| \ll |\phi_0(\eta)|$$

□ This means that we must also consider small fluctuations in the metric, because:

In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T_{\mu\nu}^{\text{inf}}(\eta, \vec{x}) = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right]$$

⇒ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G T_{\mu\nu}(x)$$

inhomogeneities of $\phi(x)$ induce inhomogeneities of $g_{\mu\nu}(x)$:

→ Consider also small inhomogeneities in the metric, i.e., in the spacetime

$$g_{\mu\nu}(x, \eta) = a(\eta) \eta_{\mu\nu} + \gamma_{\mu\nu}(x, \eta) \quad \text{with } |\gamma_{\mu\nu}(x, \eta)| \ll 1$$

- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\phi}(x)$ but this is too hard, inconsistent so far.
- In lowest order perturbation theory we first find the classical solutions $g_{\mu\nu}(\eta) = a(\eta) \eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.
- Then, we quantize only the $\hat{\gamma}(x, \eta)$ and $\hat{\gamma}_{\mu\nu}(x, \eta)$.

□ Why does this approximation help?

- * For fields $\hat{\gamma}$, $\hat{\gamma}_{\mu\nu}$ that are "small" the equations of motions are effectively linear in $\hat{\gamma}, \hat{\gamma}_{\mu\nu}$.
- * This is because we can assume that in their equations of motion all terms that are quadratic or of higher power are negligible.
- * This means that the quantum fields $\hat{\gamma}$ and $\hat{\gamma}_{\mu\nu}$ have no potential terms, nor any mass terms.
- ⇒ We will obtain a free, i.e., noninteracting quantum field theory whose nontriviality only stems from the time-varying parameters $\phi_0(\eta)$, $a_0(\eta)$.

□ Intuition: We should expect two more sources of nontriviality:

1) Interdependence of $\mathcal{C}$ and $\gamma_{\mu\nu}$ inhomogeneities:

* Much of the inhomogeneities of $\hat{\gamma}_{\mu\nu}(x, y)$ will be induced by the inhomogeneities of the inflaton, $\hat{\mathcal{C}}(x, y)$.

* Vice versa: we can also read the Einstein eqn from left to right $\Rightarrow$ these gravity inhomogeneities induce the inflaton's inhomogeneities.

* Thus, the inflaton's inhomogeneities' dynamics cannot be separated from that of the metric.

2) Some of $\gamma_{\mu\nu}(x, y)$ is independent of $\mathcal{C}$!

Recall:

Gravity is a force with some similarity to electromagnetism:

□ Some electromagnetic fields only exist because there are charges or currents.

Similarly: Some part of the metric will depend on ϕ , while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of ϕ .

□ But: also, some electromagnetic fields are self-sustaining, i.e., they exist independently, with their own dynamics.

Exercise: show this $\rightarrow$ □ Namely:

$$\vec{E}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Any vector field $\vec{E}(x)$ can be decomposed into:

$$\vec{E}(x) = \vec{E}_s(x) + \vec{E}_v(x)$$

↑
"gradient part"
or "scalar part"

↑
"curl part"
or "vector part"

Here, $\vec{E}_s$ and $\vec{E}_v$ derive from a scalar function Λ and a vector field $\vec{A}$ respectively:

$$\vec{E}_s = -\vec{\nabla}\Lambda \text{ and } \vec{E}_v = \vec{\nabla} \times \vec{A}$$

They obey: $\vec{\nabla} \times \vec{E}_s = \vec{0}$ and $\vec{\nabla} \cdot \vec{E}_v = 0$ (A)

Exercise for physics students: verify $\rightarrow$

□ According to the Maxwell equations, the scalar part, e.g., of the electric field, $\vec{E}_s$, is caused by (or causes) the electric charge density

$$\vec{\nabla} \cdot \vec{E}_s = \rho$$

⌚ An unusual but mathematically equivalent viewpoint.
* E.g. D-branes in string theory are charges that are defined from this viewpoint.

Thus, $\vec{E}$ and $\vec{B}$ fields can sustain each other, which makes possible nonzero electromagnetic fields (namely traveling waves) even where there are no charges.

while the vector part is charge independent

$$\vec{\nabla} \times \vec{E}_v = -\frac{\partial \vec{B}}{\partial t}$$

and similarly for the magnetic field $\vec{B}(x)$.

□ Similarly, some curvature exists only where there is energy-momentum.

□ But, also, some curvature is self-sustaining, with dynamics, e.g., gravitational waves.

$\Rightarrow$ We should expect that $\hat{T}_{\mu\nu}(x, y)$ contains:

* some curvature that is induced by (or induces) the inflaton inhomogeneities.

* some curvature inhomogeneities that are self-sustaining, i.e., that possess their own dynamics - and therefore also their own quantum fluctuations.

How to separate these inhomogeneities of $\gamma_{\mu\nu}(x, y)$?

Similar to vector fields $\vec{E}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ we have for the tensor field γ :

□ The perturbations $\gamma_{\mu\nu}$ of the metric tensor can be decomposed into three types:

- a) The part of $\gamma_{\mu\nu}$ which can be derived from scalar functions.
- b) The part of $\gamma_{\mu\nu}$ which can be derived from vector fields.
- c) The part of $\gamma_{\mu\nu}$ which is purely tensor.

Decomposition of $g_{\mu\nu}(x, y)$, with respect to its spatial structure:

□ One usually writes the "line element" ds^2 , i.e., the infinitesimal proper distance (squared) from x to $x+dx$ as

$$ds^2 = g_{\mu\nu}(x, y) dx^\mu dx^\nu \text{ with } dx^\mu = (dy, dx^1, dx^2, dx^3)$$

□ Then, the decomposition takes the form:

$$ds^2 = \overbrace{a^2(y) \left(dy^2 - \sum_{i=1}^3 (dx^i)^2 \right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_s^2}_{\text{scalar}} + \underbrace{ds_v^2}_{\text{vector}} + \underbrace{ds_t^2}_{\text{tensor}}$$

□ Here, the spatially "scalar" part of the inhomogeneities reads

$$ds_s^2 = a^2(y) \left[2\Phi(x, y) dy^2 - 2 \sum_{i=1}^3 \frac{\partial}{\partial x^i} B(x, y) dx^i dy - \sum_{i,j=1}^3 \left(2\Psi(x, y) \delta_{ij} - 2 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} E(x, y) \right) dx^i dx^j \right]$$

where $\Phi, \bar{\Psi}, B$ and E are scalar functions.

□ The spatially "vector" part of the metric reads:

$$ds_v^2 = a^2(\eta) \left[2 \sum_{i,j=1}^3 V_i(x, \eta) dx^i d\eta - \sum_{i,j=1}^3 \left(\frac{\partial}{\partial x^j} W_i(x, \eta) + \frac{\partial}{\partial x^i} W_j(x, \eta) \right) dx^i dx^j \right]$$

where V_i and W_i are 3-vector fields.

□ The spatially "tensor" part of the metric is the remainder, i.e., is what cannot be derived from a scalar or vector fields:

$$ds_T^2 = a^2(\eta) \sum_{i,j=1}^3 h_{ij}(x, \eta) dx^i dx^j$$

Here, h_{ij} is a spatial tensor field.

□ Remark regarding the fields V_i, W_i and h_{ij} :

Analogous to the equations (A) above in electromagnetism,

$$\vec{\nabla} \times \vec{E}_s = \vec{0} \text{ and } \vec{\nabla} \cdot \vec{E}_s = 0,$$

we now have:

* V_i, W_i obey:

$$\vec{\nabla} \cdot \vec{V} = 0, \quad \vec{\nabla} \cdot \vec{W} = 0 \quad \left(\text{i.e. } \sum_{i=1}^3 \frac{\partial}{\partial x^i} V^i = 0 \text{ etc.} \right)$$

* h_{ij} obeys:

$$h_{ij} = h_{ji}, \quad \sum_{i=1}^3 h_{ii} = 0, \quad \sum_{i=1}^3 \frac{\partial}{\partial x^i} h_{ij} = 0$$

Remark: This implies that h_{ij} describes "Weyl curvature" which is known to describe gravitational waves