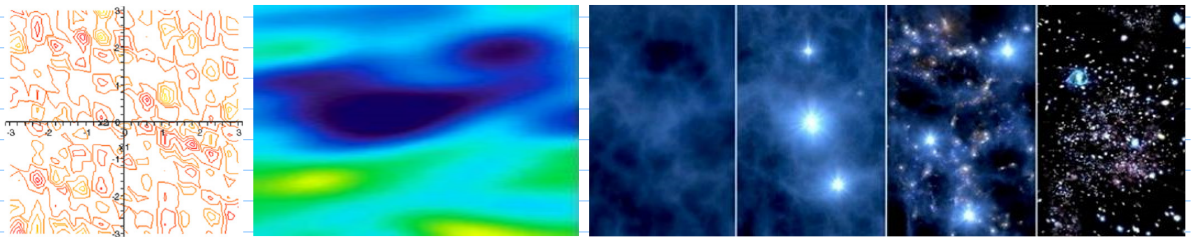
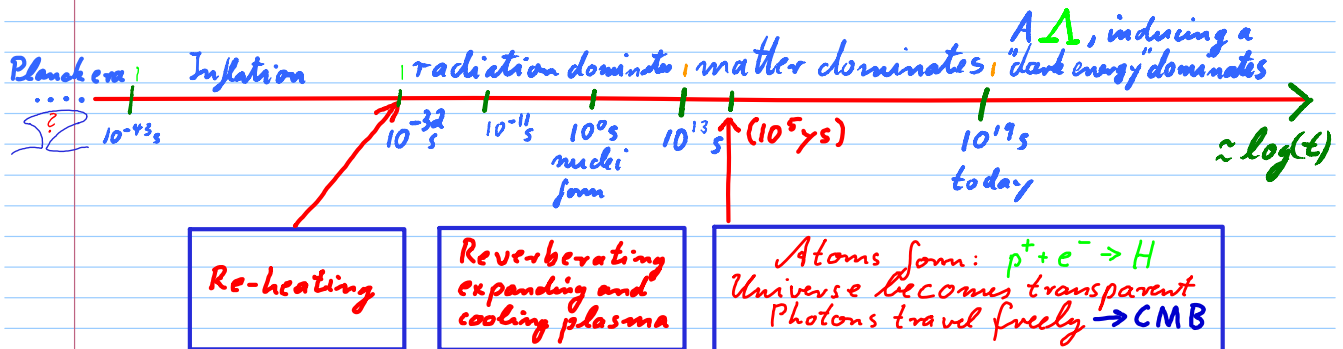
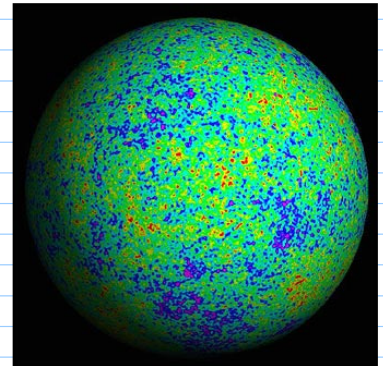
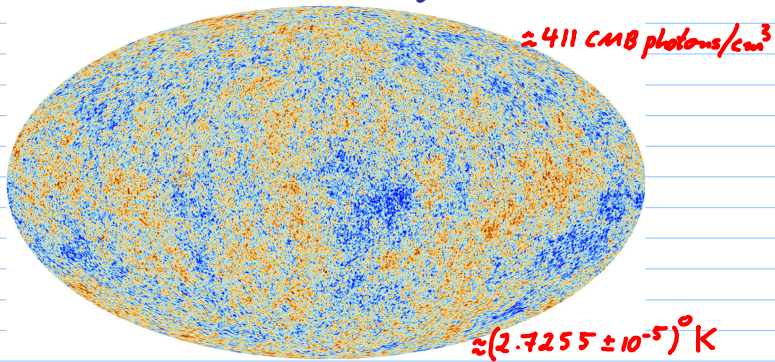


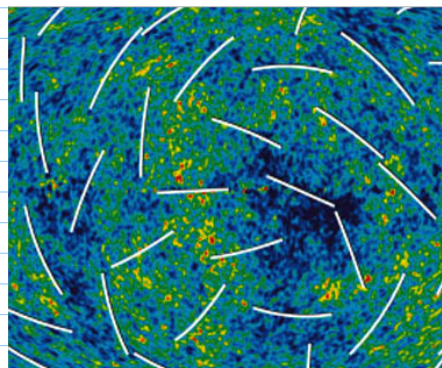
Time line of standard model of cosmology:



Actual observations of the CMB:



Zoom-in,
with polarization:
(avg polarization $\sim 10^{-6}$)



Recall:

$$\phi(x, \eta) = \phi_0(\eta) + \varphi(x, \eta) \quad \text{with } |\varphi(x, \eta)| \ll |\phi_0(\eta)|$$

$$g_{\mu\nu}(x, \eta) = a(\eta) \eta_{\mu\nu} + \gamma_{\mu\nu}(x, \eta) \quad \text{with } |\gamma_{\mu\nu}(x, \eta)| \ll 1$$

$$ds^2 = a^2(\eta) \left(d\eta^2 - \sum_{i=1}^3 (dx^i)^2 \right) + \underset{\text{scalar}}{ds_s^2} + \underset{\text{vector}}{ds_v^2} + \underset{\text{tensor}}{ds_T^2}$$

$$ds_s^2 = a^2(\eta) \left[2\Phi(x, \eta) d\eta^2 - 2 \sum_{i=1}^3 \frac{\partial}{\partial x^i} B(x, \eta) dx^i d\eta - \sum_{i,j=1}^3 \left(2\Phi(x, \eta) \delta_{ij} - 2 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} E(x, \eta) \right) dx^i dx^j \right]$$

$$ds_v^2 = a^2(\eta) \left[2 \sum_{i=1}^3 V_i(x, \eta) dx^i d\eta - \sum_{i,j=1}^3 \left(\frac{\partial}{\partial x^j} W_i(x, \eta) + \frac{\partial}{\partial x^i} W_j(x, \eta) \right) dx^i dx^j \right]$$

$$ds_T^2 = a^2(\eta) \sum_{i,j=1}^3 h_{ij}(x, \eta) dx^i dx^j$$

The surviving gauge-invariant degrees of freedom are:

▣ The purely tensorial part of the metric: $h_{ij}(x, \eta)$

▣ A combination of a scalar part of the metric, $\Phi(x, \eta)$, and $\varphi(x, \eta)$:

$$r(x, \eta) := -\frac{a_i'}{a_i} (\phi_0(\eta))^{-i} \varphi(x, \eta) - \Phi(x, \eta)$$

They possess these actions:

$$S_T = \frac{1}{64\pi G} \sum_{i,j=1}^3 \int a^2(\eta) \frac{\partial}{\partial x^\mu} (h_{ij}^{\prime}(x, \eta)) \frac{\partial}{\partial x^\nu} (h_{ij}^{\prime}(x, \eta)) \eta^{\mu\nu} d^4x$$

$$S_s = \frac{1}{2} \int z^2(\eta) \left(\frac{\partial}{\partial x^\mu} r(x, \eta) \right) \left(\frac{\partial}{\partial x^\nu} r(x, \eta) \right) \eta^{\mu\nu} d^4x \quad \text{with } z(\eta) := \frac{a_i^2(\eta)}{a_i'(\eta)} \phi_0'(\eta)$$

To quantize without a friction term, change variable:

$$u(x, \eta) := -\bar{z}(\eta) r(x, \eta)$$

convenient factors

$$p_{ij}(x, \eta) := \frac{1}{\sqrt{32\pi G}} a(\eta) h_{ij}(x, \eta)$$

Further, separate of polarization matrices:

$$p_{ij}(k, \eta) := \sum_{\lambda=1,2} v_{k,\lambda}(\eta) \varepsilon_{ij}(k, \lambda)$$

⇒ Equations of motion:

$$\hat{v}_{k,2}''(\eta) + \left(k^2 - \frac{a''}{a}\right) \hat{v}_{k,2}(\eta) = 0$$

$$\hat{u}_k''(\eta) + \left(k^2 - \frac{\bar{z}''(\eta)}{\bar{z}(\eta)}\right) \hat{u}_k(\eta) = 0$$

Quantum fluctuations

As before, this reduces to solving the eqns of motion for the mode functions, which are complex number-valued, say $\tilde{u}_k(\eta)$, $\tilde{v}_{k,2}(\eta)$:

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{\bar{z}''(\eta)}{\bar{z}(\eta)}\right) \tilde{u}_k(\eta) = 0$$

$$\tilde{v}_{k,2}''(\eta) + \left(k^2 - \frac{a''}{a}\right) \tilde{v}_{k,2}(\eta) = 0$$

along with the Wronskian conditions.

Initial conditions?

At early times:

* The k^2 term dominates

⇒ Can choose Minkowski-like init. cond.

We say we choose the "Bunch Davies vacuum".

□ The mode f at late times?

At late times:

- * The mode k crossed the Hubble horizon:
- * The terms $\frac{z''}{z}$ and $\frac{a''}{a}$ dominate.
- * The harmonic oscillator is inverted
- * Instead of 2 oscillatory basis sol's we now expect one growing and one decaying basis solution.
- * Soon after horizon crossing the mode function consists of essentially only the growing solution.

Which is the growing solution at late times?

Eqs of motion after horizon crossing:

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right) \tilde{u}_k(\eta) = 0, \text{ i.e., } \frac{\tilde{u}_k''(\eta)}{\tilde{u}_k(\eta)} = \frac{z''(\eta)}{z(\eta)}$$

$$\tilde{v}_{k,2}''(\eta) + \left(k^2 - \frac{a''}{a}\right) \tilde{v}_{k,2}(\eta) = 0, \text{ i.e., } \frac{\tilde{v}_{k,2}''(\eta)}{\tilde{v}_{k,2}(\eta)} = \frac{a''(\eta)}{a(\eta)}$$

$\Rightarrow$ Growing solution must behave as:

$$\tilde{u}_k(\eta) \sim z(\eta) \text{ at late } \eta$$

$$\tilde{v}_{k,2}(\eta) \sim a(\eta) \text{ at late } \eta$$

$\Rightarrow$ The mode f at $\tilde{r}_k(\eta) = -\frac{\tilde{u}_k(\eta)}{z(\eta)}$ and $\tilde{h}_{r,j,k}(\eta) = 32\pi G \frac{\tilde{v}_{k,2}(\eta) \epsilon_{r,j}(k,2)}{a(\eta)}$ become constant at late η , i.e., after the mode k crosses the horizon!

Check: $\tilde{v}_k(\eta) = \frac{1}{z(\eta)} \tilde{u}(\eta) \sim \frac{z(\eta)}{z(\eta)} \quad \text{for late } \eta$

$$\tilde{h}_{ij,k}(\eta) = \frac{1}{a(\eta)} \tilde{p}_{ij,k}(\eta) \sim \frac{1}{a(\eta)} \tilde{v}_{k,i,k}(\eta) \sim \frac{a(\eta)}{a(\eta)} \quad \text{for late } \eta$$

$\Rightarrow$ As expected, the magnitude of the mode k 's quantum fluctuations

$$\delta r_k = \underbrace{z^{-1} k^{3/2} |\tilde{u}_k|}_{= a^{-1} k^{3/2} |\tilde{v}|} \quad \text{and} \quad \delta h_{ij,k} = \underbrace{k^{3/2} |\tilde{h}_{ij,k}|}$$

stay constant at the value that they possess when the mode crosses the horizon, even as the mode's proper wavelength then continues to increase rapidly.

* Goal now: Calculate the magnitude of the fluctuations at horizon crossing!

Realistic example: "Power law inflation"

□ We need an explicit potential $V(\phi)$ in order to be able to find explicit $a(\eta)$, $\phi_0(\eta)$ for which to calculate then the fluctuation spectrum.

□ De Sitter is ruled out because:

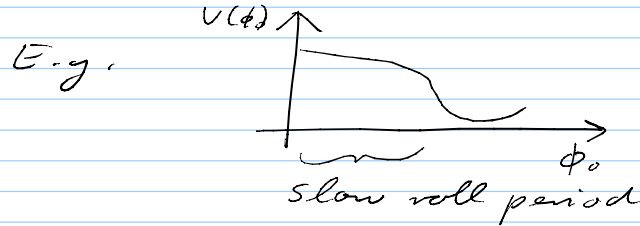
* $V(\phi)$, and therefore the temporary "cosmological constant" $H \sim \sqrt{V(\phi)}$ must slowly decrease (slow roll).

* In any case, our perturbation assumptions don't allow exact de Sitter, as δr_k would diverge, invalidating the assumption that it is small.

The "slow roll parameters"

Idea:

- * We do not know the exact slow roll potential:



- * For all values of ϕ during the inflationary period we can parametrize the slope of the potential by its derivatives.
- * These are the so-called slow roll parameters: (Recall: $H(\phi) \sim \sqrt{V(\phi)}$)

$$\varepsilon(\phi) := \overset{\text{convenience factor}}{\frac{1}{4\pi G}} \left(\frac{H'(\phi)}{H(\phi)} \right)^2 \quad \left(= \frac{\frac{3}{2} \dot{\phi}^2}{V + \frac{1}{2} \dot{\phi}^2} \right)$$

$$\eta(\phi) := \frac{1}{4\pi G} \frac{H''(\phi)}{H(\phi)} \quad \left(= \varepsilon - \frac{\varepsilon'}{\sqrt{16\pi G \varepsilon}} \right)$$

$$\xi(\phi) := \frac{1}{4\pi G} \sqrt{\frac{H'(\phi)H''(\phi)}{H^2(\phi)}}$$

etc...

□ The simplest solvable case:

- * The simplest case is that of

$$\varepsilon(\phi) = c \quad \text{where } c \text{ is a constant.}$$

* In this case:

$$c = \varepsilon(\phi) := \frac{1}{4\pi G} \left(\frac{H'(\phi)}{H(\phi)} \right)^2$$

Thus,

$$H(\phi) \sim e^{\sqrt{4\pi G c} \phi}$$

and the potential is of the form:

$$V(\phi) = e^{s\phi}$$

* Exercise: What is the value of s ?

* Then, one also finds:

$$c = \varepsilon = \eta = \xi = \dots$$

* The expansion rate is polynomial:

Exercise:

Show that:

$$a(t) = a_0 t^{1/c} \quad (t \text{ is proper time})$$

Exercise:

Show that, in terms of the conformal time η :

$$a(\eta) = \frac{-1}{\eta H} \frac{1}{1-\varepsilon}$$

Note: Still η is always negative and $t \rightarrow \infty$ means $\eta \rightarrow 0$.

The mode equations:

□ Scalar: We can now calculate $z(\eta) = \frac{a^2(\eta)}{a'(\eta)} \phi'_0(\eta)$ and therefore also the mode equation's term z''/z explicitly, to obtain

↙ A Bessel differential equation

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{(\nu^2 - 1/4)}{\eta^2} \right) \tilde{u}_k(\eta) = 0$$

where: $\nu := \frac{3}{2} + \frac{c}{1-c}$

* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\frac{\pi}{2}} (-\eta)^{\nu/2} H_\nu^{(1)}(-k\eta)$$

↑ Hankel fctn of 1st kind of order ν .

* Behavior after horizon crossing:

$$\tilde{u}_k(\eta) \rightarrow e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2}k} (-k\eta)^{-\nu+1/2}$$

* Thus, the magnitude of intrinsic curvature fluctuations after horizon crossing becomes:

$$\delta r_k(\eta > \eta_{\text{hor}}(k)) = G 2^{\nu-\frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (\nu-\frac{1}{2})^{1/2-\nu} \frac{H^2}{|H'|}$$

at $k=aH$
↑
horizon crossing

Exercise: verify

Notice: Measurement of δr_k in CMB can only tell us $\frac{H^2}{H'}$ (at horizon crossing) but not H or H' individually!

Intuition?

Earlier, for a K.G. field ϕ in a fixed background FRW universe, we found:

$$\delta\phi_k \sim H$$

Now, for the intrinsic curvature (the Mukhanov variable), we found:

$$\delta r_k \sim H^2/|H'|.$$

Recall: r_k is the slicing-independent combination of the scalar part of

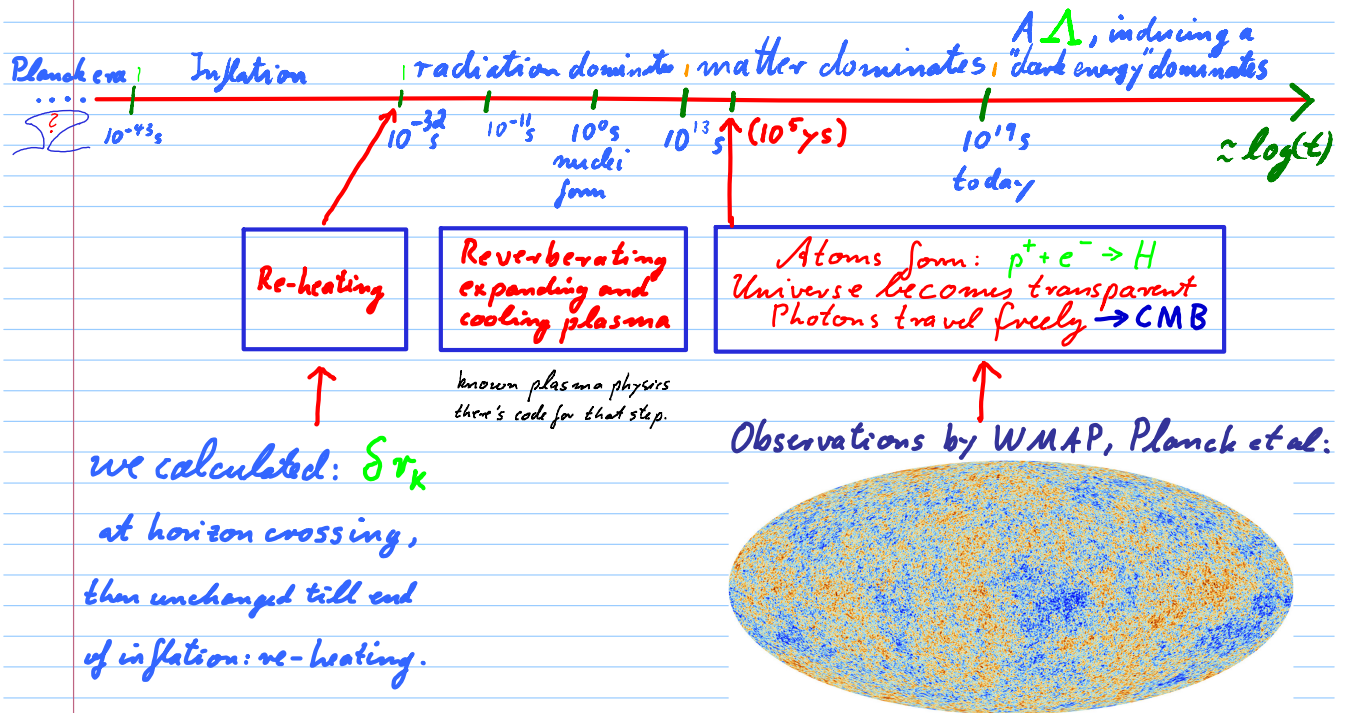
$\delta g_{\mu\nu}$ and ϕ .

The slower the roll ($|H'|$ small) the wider away from another fluctuate gauge equivalent and inequivalent slicings:

Analogous to: A river in a plain meanders the more widely, the flatter the plain is.

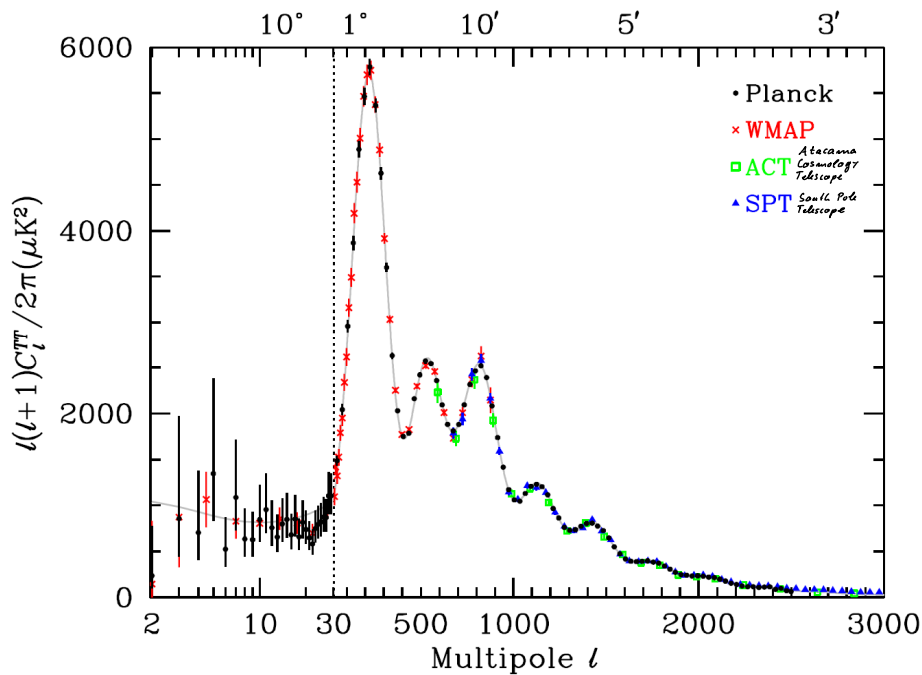


Recall the timeline:



- δr_k is predicted to have seeded oscillations in the hot plasma after re-heating. The plasma decohered the quantum fluctuations of the intrinsic curvature r .
- Standard plasma physics allows one to calculate the propagation and dispersion for the $\approx 10^5ys$ until hydrogen formed.
- The temperature fluctuation spectrum in the CMB is from gravitational blue and redshifts due to these curvature fluctuations.
- Theory matches experiment closely, while fixing cosmological parameters, including indications that $\epsilon \neq 0$, namely that $\delta r_k \neq \text{const.}$

Best fit today:



$$K = 0$$

$$\Omega \approx 0.7 \rho_{\text{critical}}$$

$$\rho_{\text{matter}} \approx 0.3 \rho_{\text{critical}}$$

$$\rho_{\text{dark matter}} \approx 0.9 \rho_{\text{matter}}$$

$$\rho_{\text{visible matter}} \approx 0.1 \rho_{\text{matter}}$$

$$\rho_{\text{curv}} \approx 5 \cdot 10^{-5} \rho_{\text{critical}}$$

$$v_{\text{peculiar}} \approx 370 \text{ km/s of earth}$$

▮ Tensor modes: $\ddot{\tilde{v}}_{k,i} + \left(k^2 - \frac{a''}{a}\right) \tilde{v}_{k,i} = 0$

we obtain for the term a''/a :

$$\frac{a''}{a} = 2a^2 H^2 \left(1 - \epsilon/2\right)$$

which comes out to be (verify):

$$\frac{a''}{a} = \frac{1}{\eta^2} \left(v^2 - \frac{1}{4}\right) \quad \text{recall: } v = \frac{3}{2} - \frac{c}{1-c}$$

⇒ The mode eqn is again solved by the Hankel functions.

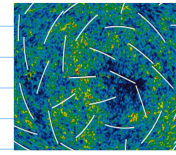
⇒ The tensor fluctuation spectrum:

$$\delta h_{ij,k} = \frac{2}{\sqrt{\pi}} 2^{v-\frac{1}{2}} \frac{\Gamma(v)}{\Gamma(3/2)} (v-1/2)^{v-1/2} \sqrt{G} H \Big|_{k=aH}$$

horizon crossing.
↓

δh_{ij} should have left curl ("B") polarization in the CMB

Experiments show polarization in the CMB:



□ But most is gradient ("E") polarization that originated in δv_n or in foreground.

□ So far, h_{ij} -originated B-polarization cannot be distinguished from foreground. see BICEP2 (2014)

□ Observation of h_{ij} polarization:

* Would show quantized gravitational waves!

* Would determine the scale of H , and therefore of H' !

* This would tell the slope of the spectra

$\Rightarrow$ Nontrivial consistency conditions to check inflation.