

The driven harmonic oscillator cont'd:

D. Energy eigenstates

* Recall $\hat{H}(t) = \omega (a^\dagger(t) a(t) + \frac{1}{2}) - \frac{1}{\sqrt{2\omega}} (a^\dagger(t) + a(t)) J(t)$

$$= \begin{cases} \omega (a_{in}^\dagger a_{in} + \frac{1}{2}) & \text{for } t < 0 \\ \text{something} & \text{for } 0 \leq t \leq T \\ \omega (a_{out}^\dagger a_{out} + \frac{1}{2}) & \text{for } T < t \end{cases}$$

Here, $a_{in} := a(0)$, $a_{out} := a(T)$ and $a_{out} = a_{in} + J_0$

with: $J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

* For $t < 0$, we diagonalized the Hamiltonian

$$\hat{H}(t) = \omega (a_{in}^\dagger a_{in} + \frac{1}{2}) = \hat{H}_{in} = \text{const.}$$

by using $[a_{in}, a_{in}^\dagger] = 1$ to construct its eigenbasis:

$$\hat{H}_{in} |n_{in}\rangle = E_n^{(in)} |n_{in}\rangle$$

Namely:

$$E_n^{(in)} = \omega (n + \frac{1}{2}), \quad n = 0, 1, 2, 3, \dots$$

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}} (a_{in}^\dagger)^n |0_{in}\rangle$$

Note: The set $\{|n_{in}\rangle\}$ is a Hilbert basis of the Hilbert space $\mathcal{H}$.

* By $t > T$, the Hamiltonian has become a different operator:

$$\hat{H}(t) = \omega \left(a_{out}^+ a_{out} + \frac{1}{2} \right) = \hat{H}_{t>T} = \text{const.}$$

What are its eigenvectors $|n_{out}\rangle$ and eigenvalues E_n^{out} ?

Observation:

We have:

$$[a_{out}, a_{out}^+] = 1$$

$\Rightarrow$ we can construct the eigenbasis of $H_{t>T}$

with the same method as the eigenbasis of $H_{t<0}$:

* There is a unique vector $|0_{out}\rangle \in \mathcal{H}$ obeying:

$$a_{out} |0_{out}\rangle = 0$$

* We define the set of vectors $\{|n_{out}\rangle\}$:

$$|n_{out}\rangle := \frac{1}{\sqrt{n!}} (a_{out}^+)^n |0_{out}\rangle$$

* Proposition:

$$\hat{H}_{t>T} |n_{out}\rangle = E_n^{(out)} |n_{out}\rangle \text{ with } E_n^{(out)} = \omega \left(n + \frac{1}{2} \right) = E_n^{(in)}$$

The operators $\hat{H}_{t<0}$ and $\hat{H}_{t>T}$ are different and have different eigenvectors: $|n_{in}\rangle$ and $|n_{out}\rangle$. Why are the eigenvalues the same? They both describe a free oscillator of frequency ω .

* Proposition:

The set $\{|n_{out}\rangle\}$ is a ON Hilbert basis of the Hilbert space $\mathcal{H}$.

How are the two bases related?

* Recall: Both, $\{|n_{in}\rangle\}$ and $\{|n_{out}\rangle\}$ are ON bases of $\mathcal{H}$.

$\Rightarrow$ Each basis vector $|n_{in}\rangle$ is a linear combination of the basis vectors $\{|n_{out}\rangle\}$ and vice versa.

* Therefore, in particular:

There must exist coefficients $\Lambda_n \in \mathbb{C}$ so that:

$$|0_{in}\rangle = \sum_n \Lambda_n |n_{out}\rangle$$

$\uparrow$ "Bogolubov Transformation"

* Meaning of the Λ_n ?

□ The system is frozen in state $|x\rangle = |0_{in}\rangle$.

□ Assume we measure at a time $t > T$ the energy, i.e., we measure

$$\hat{H}(t) = \omega (a_{out}^\dagger a_{out} + \frac{1}{2})$$

□ What is the probability amplitude for finding the energy eigenvalue E_n ?

□ Clearly:

$$\text{prob. amp. } (|n_{out}\rangle \text{ at } t > T) = \langle n_{out} | x \rangle$$

$$\text{i.e.:} \quad \text{prob. } (|n_{out}\rangle \text{ at } t > T) = |\langle n_{out} | x \rangle|^2$$

□ Calculate:

$$\begin{aligned}\langle n_{out} | \psi \rangle &= \langle n_{out} | 0_{in} \rangle \\ &= \langle n_{out} | \sum_m \Lambda_m | m_{out} \rangle \\ &= \Lambda_n\end{aligned}$$

⇒ If the oscillator started in its ground state, then at time $t > T$ the probability for finding the oscillator in its n 'th excited state is given by:

$$\text{prob.}(|n_{out}\rangle \text{ at } t > T) = |\Lambda_n|^2$$

Remark: In QFT, this will be the prob. for finding n particles after charges and currents $J(x,t)$ excited the vacuum.

Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n$

Proof: The claim is that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$.

We need to check that indeed: $a_{in} |0_{in}\rangle = 0$

Using $a_{out} = a_{in} + J_0$, we need to check: $(a_{out} - J_0) |0_{in}\rangle = 0$

Indeed:

$$\begin{aligned}(a_{out} - J_0) \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle \\ = e^{-\frac{1}{2}|J_0|^2} (a_{out} - J_0) \sum_n J_0^n \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n!}} (a_{out}^\dagger)^n |0_{out}\rangle\end{aligned}$$

$$\begin{aligned}
&= e^{-\frac{i}{2}|\gamma_0|^2} (a_{out} - \gamma_0) e^{i\gamma_0 a_{out}^\dagger} |0_{out}\rangle \\
&= e^{-\frac{i}{2}|\gamma_0|^2} \left(\underbrace{a_{out} e^{i\gamma_0 a_{out}^\dagger}}_{\text{using } AB = [A, B] + BA} - \gamma_0 e^{i\gamma_0 a_{out}^\dagger} \right) |0_{out}\rangle \\
&= e^{-\frac{i}{2}|\gamma_0|^2} \left(\underbrace{[a_{out}, e^{i\gamma_0 a_{out}^\dagger}] + e^{i\gamma_0 a_{out}^\dagger} a_{out}}_{=} - \gamma_0 e^{i\gamma_0 a_{out}^\dagger} \right) |0_{out}\rangle \\
&\stackrel{(*)}{=} e^{-\frac{i}{2}|\gamma_0|^2} \left((\gamma_0 - \gamma_0) e^{i\gamma_0 a_{out}^\dagger} + e^{i\gamma_0 a_{out}^\dagger} a_{out} \right) |0_{out}\rangle = 0 \quad \checkmark
\end{aligned}$$

Note: In the last step, $(*)$, we used that: $[a_{out}, e^{i\gamma_0 a_{out}^\dagger}] = \gamma_0 e^{i\gamma_0 a_{out}^\dagger}$.

Exercise: Show that, more generally, $[a, a^\dagger] = 1$ implies $[a, f(a^\dagger)] = f'(a^\dagger)$ by induction.

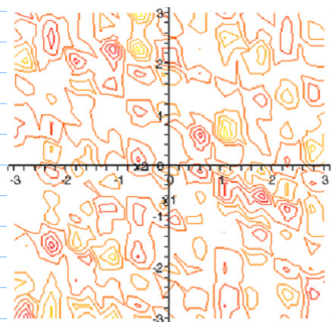
Hint: Show that: $[a, a^\dagger] = 1, [a, a^{\dagger 2}] = 2a^\dagger, [a, a^{\dagger 3}] = 3a^{\dagger 2}, \dots, [a, a^{\dagger n}] = n(a^\dagger)^{n-1}$

Exercise: Verify that $|0_{in}\rangle = \sum_n e^{-\frac{i}{2}|\gamma_0|^2} \frac{1}{n!} \gamma_0^n |n_{out}\rangle$ obeys $\langle 0_{in} | 0_{in} \rangle = 1$.

Drive mode oscillators in QFT:

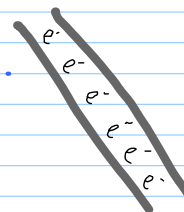


Making waves...



Making EM waves...

e^+



Recall:

$$\hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + j(x,t)\hat{\phi}(x,t) d^3x$$

Example interpretation:

* $\hat{\phi}(x,t)$ may be viewed as a slightly simplified version of the quantum electromagnetic field.

* $j(x,t)$ may be viewed as a simplified version of a given classical electric charge and current density functions.

Example:

A (Klein - Gordon) charge traveling a path $\tilde{x}^i(t)$:

Then: $j(x,t) = q \delta(x - \tilde{x}(t))$

In- and out periods

Let us consider the case where

$$j(x,t) = 0 \text{ for all } t \notin [0, T]$$

$\Rightarrow$ It suffices to consider the periods $t < 0$ and $t > T$ in both of which $j(x,t) = 0$ (and then to relate the bases).

The free (i.e., undriven) QFT: ($t < 0 \sim t > T$)

$$\hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) d^3x$$

* We need to solve:

$$\hat{\Pi}(x,t) - (\Delta - m^2) \hat{\phi}(x,t) = 0$$

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = i \delta^3(x-x')$$

* Fourier transformed,

$$\hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\phi}(x,t) e^{-ikx} d^3x$$

we need to solve:

$$\ddot{\hat{\phi}}_k(t) + \underbrace{(k^2 + m^2)}_{\substack{= \sum_{i=1}^3 k_i^2 \\ \text{Definition: } =: \omega_k^2}} \hat{\phi}_k(t) = 0 \quad (EoM)$$

$$[\hat{\phi}_k(t), \hat{\Pi}_{k'}(t)] = i \delta^3(k+k') \quad (CCRs)$$

* Recall: $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$ means $\hat{\phi}_k^\dagger(t) = \hat{\phi}_{-k}(t)$.

Solution strategy due to Fock:

* Proceed analogously to the driven oscillator, e.g., during $t < 0$:

□ Introduce new variables:

$$QM: \quad a(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$

$$QFT: \quad a_k(t) := \sqrt{\frac{\omega_k}{2}} \hat{\phi}_k(t) + i \frac{1}{\sqrt{2\omega_k}} \hat{\Pi}_k(t)$$

□ Equation of motion and CCRs:

$$QM: \quad \dot{a}(t) = -i\omega a(t) \quad [a(t), a^\dagger(t)] = 1$$

$$QFT: \quad \dot{a}_k(t) = -i\omega_k a_k(t) \quad [a_k(t), a_{k'}^\dagger(t)] = \delta^3(k-k') \quad \text{Exercise: verify}$$

□ Remark: Valid only while no force and while ω is constant.

▢ Solution, using an initial condition:

QM: $a(t) = e^{-i\omega t} a_{in}, \quad [a_{in}, a_{in}^\dagger] = 1$

QFT: $a_k(t) = e^{-i\omega_k t} a_{in_k}, \quad [a_{in_k}, a_{in_k'}^\dagger] = \delta^3(k-k')$

▢ Explicitly $\Rightarrow$

QM: $\hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^\dagger \right)$

QFT: $\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in_k}^\dagger \right) \quad (S)$

Exercise:
verify $\rightarrow$

(i.e.: $\hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in_k} e^{-i\omega_k t + ikx} + a_{in_k}^\dagger e^{i\omega_k t - ikx} \right) d^3k$)

The Hilbert space of states:

* Analogous to the case of QM, there is a vector, $|0_{in}\rangle \in \mathcal{H}$, which obeys:

$a_{in_k} |0_{in}\rangle = 0$, now for all vectors k .
 $(|0_{in}\rangle = \bigotimes_k |0_{in_k}\rangle)$

* The Hamiltonian reads (for $t < 0$):

Exercise: $\rightarrow$ verify $\rightarrow$

$$\begin{aligned} \hat{H} &= \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}_k^\dagger \hat{\pi}_k + \omega^2 \hat{\phi}_k^\dagger \hat{\phi}_k d^3k \\ &= \int_{\mathbb{R}^3} \omega_k \left(a_{in_k}^\dagger a_{in_k} + \frac{1}{2} \delta^3(0) \right) d^3k \end{aligned}$$

$\nwarrow$ This ∞ is called a "Infrared divergence"

In a box: $\hat{H} = L^{-3/2} \sum_k \omega_k \left(a_{in_k}^\dagger a_{in_k} + \frac{1}{2} \right)$ (because $\delta(k,k')$ is now $\delta_{k,k'}$)

Notice: The divergence $\sum_k L^{-3/2} \omega_k \frac{1}{2} = \infty$ is an "ultraviolet divergence".

After the driving ends, $t > T$:

* One obtains: $a_k(t) = e^{-i\omega_k t} a_{out,k}$ with $a_{out,k} = a_{in,k} + J_{o,k}$

$$J_{o,k} := \frac{i}{\sqrt{2\omega}} \int_0^T J_k(t') e^{i\omega_k t'} dt'$$

Here: $J_k(t)$ is the Fourier transform of $J(x,t)$.

* Construct the out-basis $\{|n_{out,k}\rangle\}$ from:

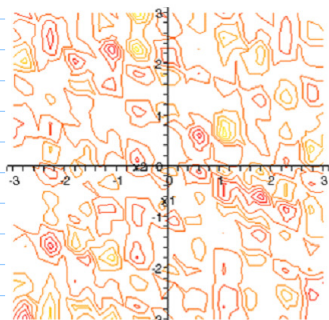
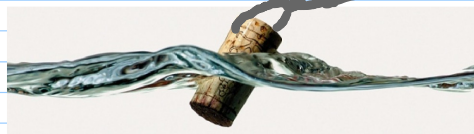
$$a_{out,k} |0_{out}\rangle = 0$$

~> Can calculate, e.g., $|\langle n_{out,k} | 0_{in} \rangle|^2$ i.e., the probability for $J(x,t)$ to have created n particles of momentum k .

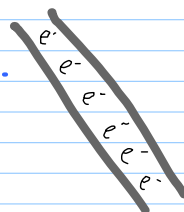
Recall:



Making waves...

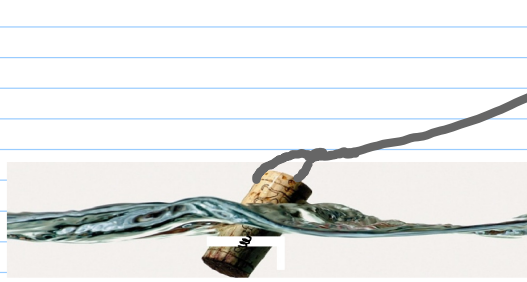
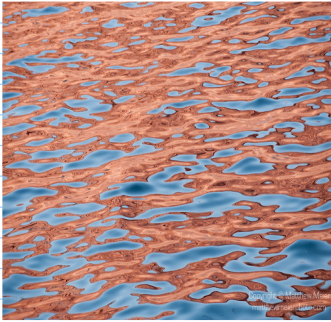


Making EM waves...

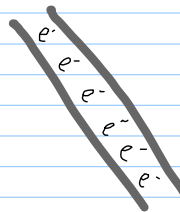
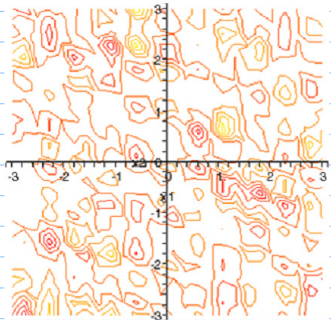


e^+
↑
Described by $J(x,t)$.

Upgrade: Give the charge $j(x,t)$ its own dynamics:



A cork with a spring.



Often, only one atomic transition is of interest. Then, we can model the atom as a 2-level system.

Described by QM, e.g. atom or qubit

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + \overset{\text{number-valued, classical}}{j(x,t)}\hat{\phi}(x,t) d^3x$$

and upgrade it to:

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} 1 \otimes \left(\hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) \right) + \hat{j}(x,t) \otimes \hat{\phi}(x,t) d^3x$$

an operator acting on the Hilbert space of the atom.

with $\hat{j}(x,t) = \hat{Q}(t) \delta(x - \tilde{x}(t))$

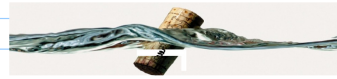
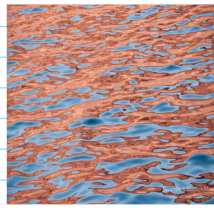
The Hilbert space: $\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{atom}} \otimes \mathcal{H}_{\text{field}}$

Simplified notation:

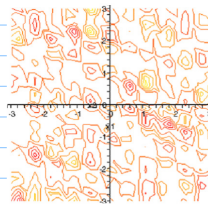
$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + \hat{j}(x,t)\hat{\phi}(x,t) d^3x$$



The charged systems can act as emitters and as receivers of waves:



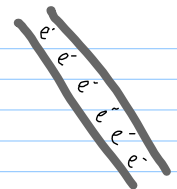
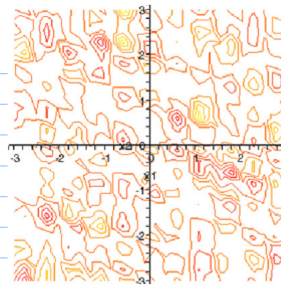
And, quantumly, they can act as emitters and receivers of particles!



Definition: (Unruh, deWitt):

A "particle", such as a photon is what a "particle detector", such as an atom, can detect, by getting excited.

With acceleration:



An accelerated atom's charges can excite the field.

This, in turn can excite the atom: the Unruh effect

→ An accelerated atom may detect particles even if inertial observers only see the vacuum

→ Related to gravity via the equivalence principle

→ Related to Hawking radiation.