

Mathematical preparations for QFT in curved space:

Plan today:

- Functional derivatives $\frac{\delta F[g]}{\delta g(x)} = ?$
- Example use 1: to make the QFT Schrödinger equation well defined.
- Example use 2: to define the Functional Legendre transform.
- Use both to obtain the Lagrangian formulation of QFT - which will be starting point for QFT on curved space.

Functional differentiation

Recall:

a.) Differentiation of functions of one variable, $F(u)$:

$$\frac{dF(u)}{du} := \lim_{\varepsilon \rightarrow 0} \frac{F(u+\varepsilon) - F(u)}{\varepsilon}$$

b.) Differentiation of functions of countably many variables, $F(\{u_j\}_{j=1,2,3,\dots})$:

$$\begin{aligned} \frac{\partial F(\{u_j\}_{j=1,2,\dots})}{\partial u_i} &:= \lim_{\varepsilon \rightarrow 0} \frac{F(u_1, \dots, u_i + \varepsilon, \dots) - F(u_1, \dots, u_i, \dots)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{F(\{u_j + \varepsilon \delta_{ij}\}_{j=1,\dots}) - F(\{u_j\}_{j=1,\dots})}{\varepsilon} \end{aligned}$$

Definition:

c.) Differentiation of functions of uncountably many

variables, $F(\{u(x)\}_{x \in \mathbb{R}^n})$:

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x \in \mathbb{R}^n})}{\delta u(y)} := \lim_{\varepsilon \rightarrow 0} \frac{F(\{u(x) + \varepsilon \delta(x-y)\}_{x \in \mathbb{R}^n}) - F(\{u(x)\}_{x \in \mathbb{R}^n})}{\varepsilon}$$

→ Since F is a "functional", i.e., is mapping functions to numbers

$$F: u \rightarrow F[u] \in \mathbb{C}$$

↑
function

↑
short for $\{u(x)\}_{x \in \mathbb{R}^n}$

we call $\frac{\delta F}{\delta u(x)}$ a functional derivative.

Example:

$$F[u] := \int_{\mathbb{R}} \cos(x) u(x)^2 dx$$

Then:

$$\frac{\delta F}{\delta u(y)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\int_{\mathbb{R}} \cos(x) (u(x) + \varepsilon \delta(x-y))^2 - \int_{\mathbb{R}} \cos(x) u(x)^2 dx \right]$$

Distribution theory would be needed. But it drops out anyway

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \cos(x) \left(u(x)^2 + \varepsilon 2u(x)\delta(x-y) + \varepsilon^2 \delta^2(x-y) - u(x)^2 \right) dx$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} 2u(x) \delta(x-y) \cos(x) dx$$

$$= 2 \cos(y) u(y)$$

Similarly, one obtains: $\frac{\delta}{\delta u(y)} \int_{\mathbb{R}} f(x) u(x)^n dx = f(y) n u(y)^{n-1}$

$\Rightarrow$ Functional derivatives act on polynomials (and suitable power series) in u by removing the integral and reducing the power in u by one, as expected from ordinary derivatives.

Remark: * Worked with $u(x)$.

* Would obtain same result if we used any other continuous or discrete basis of L^2 .

* E.g. other basis (continuous): e^{ixp} , i.e. use $\tilde{u}(p)$

* E.g. other basis (countable): $H_n(x) e^{-x^2}$, i.e. use $\tilde{u}_n$
 $\uparrow$ Hermite polynomials

$\Rightarrow$ Functional differentiation is, up to basis change, usual differentiation

Note: How can $L^2[\mathbb{R}]$ have countable basis? Recall: $L^2[\mathbb{R}]$ consists not of functions, but of equivalence classes of functions.

Example application 1:

Schrödinger equation of QFT now well defined:

QM: $\hat{q}_i$ $\hat{p}_i$ i t

QFT: $\hat{\phi}(x)$ $\hat{\pi}(x)$ x t

QM: $\hat{H}(t) = \sum_{i=1}^n \frac{\hat{p}_i^2}{2} + V(\hat{q}, t)$
 $\uparrow$ all q_i

Plays rôle of $V(\hat{q}, t)$ although the first term is usually not considered to be part of the QFT's potential.

QFT: $\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{\pi}(x)^2}{2} + \frac{1}{2} \hat{\phi}(x) (m^2 - \Delta) \hat{\phi}(x) + W(\hat{\phi}, t) d^3x$

$\uparrow$ Example: $W(\hat{\phi}) = \frac{1}{4!} \hat{\phi}^4(x, t)$
 In general: $W(\hat{\phi})$ also contains other fields

QM: Example of complete set of commuting s.adj. operators: $\{\hat{q}_i\}_{i=1}^n$

QFT: Example of complete set of commuting s.adj. operators: $\{\hat{\phi}(x)\}_{x \in \mathbb{R}^3}$

QM: The joint eigenbasis $\{|\{q_i\}_{i=1}^m\rangle\}$ of the $\{\hat{q}_i\}_{i=1}^m$ obeys:

$$\hat{q}_i |\{q_i\}_{i=1}^m\rangle = q_i |\{q_i\}_{i=1}^m\rangle$$

QFT: The joint eigenbasis $\{|\{\phi(x)\}_{x \in \mathbb{R}^3}\rangle\}$ of the $\{\hat{\phi}(x)\}_{x \in \mathbb{R}^3}$ obeys:

$$\hat{\phi}(y) |\{\phi(x)\}_{x \in \mathbb{R}^3}\rangle = \phi(y) |\{\phi(x)\}_{x \in \mathbb{R}^3}\rangle$$

QM: Wave function of a state $|\psi(t)\rangle \in \mathcal{H}$ in position eigenbasis:

$$\psi(\{q_i\}_{i=1}^m, t) = \langle \{q_i\}_{i=1}^m | \psi(t) \rangle \quad (\text{like } \psi(q) = \langle q | \psi \rangle)$$

QFT: Wave functional of a state $|\Psi(t)\rangle \in \mathcal{K}$ in field eigenbasis: ↖ Hilbert space of QFT, of course

$$\Psi[\{\phi(x)\}_{x \in \mathbb{R}^3}, t] = \langle \{\phi(x)\}_{x \in \mathbb{R}^3} | \Psi(t) \rangle$$

↑ Probability amplitude for finding function $\phi(x)$ when measuring $\hat{\phi}(x)$ at t .

Simplified notation:

QM: $\psi(q, t) = \langle q | \psi(t) \rangle$

QFT: $\Psi[\phi, t] = \langle \phi | \Psi(t) \rangle$

QM: Representation of $\hat{q}_i, \hat{p}_i$ obeying $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ in $\hat{q}$ eigenbasis:

$$\hat{q}_j: \psi(q, t) \rightarrow q_j \psi(q, t)$$

$$\hat{p}_j: \psi(q, t) \rightarrow -i \frac{\partial}{\partial q_j} \psi(q, t)$$

QFT: Representation of $\hat{\phi}(x), \hat{\pi}(y)$ obeying $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x-y)$ in $\hat{\phi}$ eigenbasis:

$$\hat{\phi}(x): \Psi[\phi, t] \rightarrow \phi(x) \Psi[\phi, t]$$

$$\hat{\pi}(x): \Psi[\phi, t] \rightarrow -i \frac{\delta}{\delta \phi(x)} \Psi[\phi, t]$$

Exercise:

Verify that $\hat{\phi}(x), \hat{\pi}(x)$ obey the CCRs.

QM: Schrödinger equation:

$$i \frac{d}{dt} \Psi(q, t) = \sum_{j=1}^n -\frac{1}{2} \frac{\partial^2}{\partial q_j^2} \Psi(q, t) + V(q, t) \Psi(q, t)$$

Recall: It is to be solved for all q

QFT: Schrödinger equation:

$$i \frac{d}{dt} \Psi[\phi, t] = \int_{\mathbb{R}^3} \left(-\frac{1}{2} \frac{\delta^2}{\delta \phi(x)} + \frac{1}{2} \phi(x) (m^2 - \Delta) \phi(x) + W(\phi(x), t) \right) \Psi[\phi, t]$$

Recall: It is to be solved for all ϕ

Remark: With W it can be solved only perturbatively.

Exercise: Set $W=0$. Fourier transform to k variables in box regularization. Verify that the wave functional Ψ_0 of the vacuum state obtained before does obey the Schr. eqn.

Example application 2: The functional Legendre transform!

□ Motivation? We will need to determine in curved space:

What becomes of: $\hat{\pi}(x, t) = \dot{\hat{\phi}}(x, t)$?

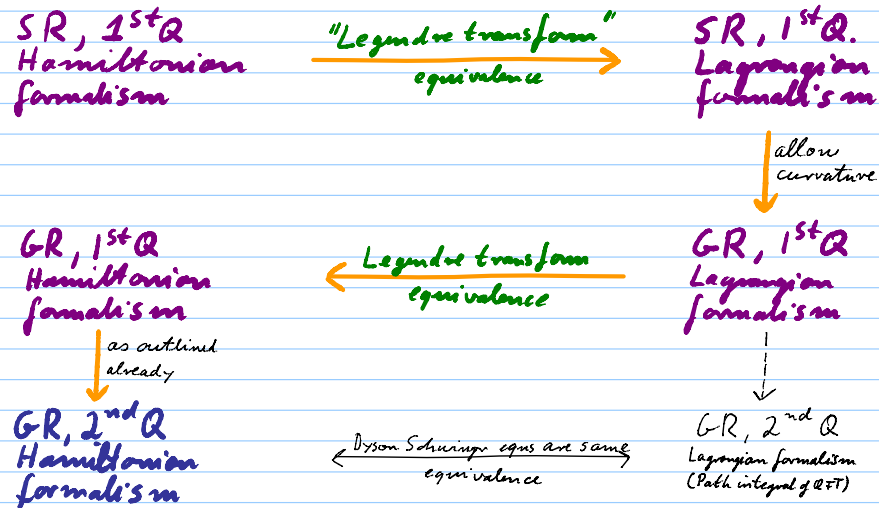
□ Problem? Time is preferred coordinate in Hamiltonian formalism.

* But the formalism must be coordinate system independent to fit general relativity (G.R).

* Now, for example, $\hat{\pi}(x, t) = \frac{d}{dt} \hat{\phi}(x, t)$ is not the same as $\hat{\pi}(x, \tau) = \frac{d}{d\tau} \hat{\phi}(x, \tau)$ for arbitrary $\tau(t)$:

$$\hat{\pi}(x, \tau) = \frac{d}{d\tau} \hat{\phi}(x, \tau(t)) = \frac{d}{d\tau} \hat{\phi}(x, \tau(t)) \left(\frac{d\tau}{dt} \right) \neq \frac{d}{dt} \hat{\phi}(x, \tau)$$

- Strategy:
1. Transform to coordinate-independent Lagrange formalism.
 2. Move from special to general relativity (GR).
 3. Transform GR result back to Hamilton formalism.
 4. Apply 2nd quantization.



The Legendre transform (LT):

- Assume given a function, $F(u)$.



- Define a new variable $w(u)$:

$$w(u) := \frac{dF}{du} \quad (\text{I})$$

- Assume that (I) can be solved to obtain:

$$u(w)$$

(that's ok if F is convex, say $F''(u) > 0$ for all u)

- The Legendre transform of F is a new function, G , of w :

$$F(u) \xrightarrow{\text{LT}} G(w)$$

- Namely: $G(w) := w u(w) - F(u(w))$

Proposition:

$$(LT)^2 = id$$

Proof:

□ Define a new variable: $v(w) := \frac{\partial G(w)}{\partial w}$

□ In fact:

$$\begin{aligned} v(w) &= \frac{\partial}{\partial w} (w u(w) - F(u(w))) \\ &= u(w) + w \frac{\partial u(w)}{\partial w} - \underbrace{\frac{\partial F(u(w))}{\partial u}}_u \frac{\partial u(w)}{\partial w} \\ &= u! \end{aligned}$$

□ Therefore LT^2 yields $F(u) \xrightarrow{LT} G(w) \xrightarrow{LT} H(v)$ with:

$$H = vw - G = \underbrace{vw}_{u \text{ from just above}} - (wu - F) = F \quad \checkmark$$

Example:

* Consider $f(a, b, c) := a e^{bc}$

* Find LT with respect to b (i.e. while treating a, c as "spectator variables"):

$$f(a, b, c) \xrightarrow[b \rightarrow \beta]{LT} g(a, \beta, c)$$

* Define $\beta(a, b, c) := \frac{\partial f}{\partial b} = a c e^{bc}$

* Invert: $b(a, \beta, c) = \frac{1}{c} \ln \frac{\beta}{ac}$

* Legendre transform: $f(a, b, c) \xrightarrow{LT} g(a, \beta, c)$

$$g(a, \beta, c) := \beta b(a, \beta, c) - f(a, b(a, \beta, c), c)$$

$$g(a, \beta, c) = \frac{\beta}{c} \ln \frac{\beta}{ac} - a e^{\frac{\beta}{c} \ln \frac{\beta}{ac}} = \frac{\beta}{c} \ln \frac{\beta}{ac} - \frac{\beta}{c}$$

Case of countably many variables:

□ How to define

$$F(\{u_i\}) \xrightarrow{LT} G(\{w_i\}) ?$$

□ Define: $w_i := \frac{\partial F}{\partial u_i}$

□ Assume we can invert to obtain:

$$u_i(\{w_i\})$$

□ Define:

$$G(\{w_i\}) := \sum_i w_i u_i(\{w_i\}) - F(\{u_i(\{w_i\})\})$$

(we may also allow for spectator variables)

Case of uncountably many variables:

□ How to define

$$F[\{u(x)\}_{x \in \mathbb{R}^n}] \xrightarrow{LT} G[\{w(x)\}_{x \in \mathbb{R}^n}] ?$$

□ Define:

$$w(x) := \frac{\delta F}{\delta u(x)}$$

□ Assume we can solve to obtain:

$$u(x, \{w(x')\}_{x' \in \mathbb{R}^n})$$

□ Define:

$$G[\{w(x)\}_{x \in \mathbb{R}^n}] := \int_{\mathbb{R}^n} w(x) u(x, \{w(x')\}_{x' \in \mathbb{R}^n}) dx - F[\{u(x, \{w(x')\})\}]$$

□ Note: We still have that $LT \circ LT = \text{id}$.

← classical mechanics

Application to CM:

* Assume the Hamiltonian $H(q, p)$ is given.

* Hamilton equations for arbitrary $f(q, p)$: Recall: Poisson bracket
 $\{q, p\} = 1$

$$\dot{f}(q, p) = \{f(q, p), H(q, p)\}$$

See my notes to AMATH 673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations unchanged while allowing q, p noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be $\hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$ and $\{\hat{f}, \hat{g}\} = \frac{1}{i\hbar} [\hat{f}, \hat{g}]$

* From this, one can prove the eqns of motion for q, p :

$$\dot{q} = \frac{\partial H(q, p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q, p)}{\partial q} \quad (\text{EoM})$$

* Legendre transform:

$$H(q, p) \xrightarrow{\text{LT}} L(q, b) \quad \begin{array}{l} \text{The "Lagrangian"} \\ (q \text{ is spectator}) \end{array}$$

* Example: $H(q, p) := \frac{p^2}{2} + V(q)$.

$$\text{Then: } b \stackrel{\text{L.T.}}{:=} \frac{\partial H(q, p)}{\partial p} \stackrel{\text{EoM}}{=} \dot{q}$$

$$\Rightarrow L(q, b) = b p(q, b) - H(q, p(q, b)) = \dot{q} p(q, \dot{q}) - H(q, p(q, \dot{q})) = L(q, \dot{q})$$

Proposition:

The equations of motion (EoM) now take the form:

$$b = \dot{q} \quad \text{and} \quad \frac{\partial L}{\partial q} = \frac{d}{dt} \frac{dL}{db} \quad (\text{Euler Lagrange equation})$$

Proof: Exercise

$$\text{Example: } H = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2 \xrightarrow{\text{LT}} L[q, b] = \frac{1}{2} \dot{q}^2 - \frac{\omega^2}{2} q^2$$

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -\omega^2 q \quad \quad -\omega^2 q = \ddot{q}, \quad b = \dot{q}$$

↖ classical (not conformal) field theory

Application to CFT:

□ Assume Hamiltonian $H(\phi, \pi)$ is given.

□ Hamilton equation for arbitrary $f(\phi, \pi)$:

$$\dot{f}(\phi, \pi, x, t) = \{f(\phi, \pi, x, t), H(\phi, \pi)\}$$

$$\text{with: } \{\phi(x, t), \pi(x', t)\} = \delta^3(x - x')$$

□ This yields the eqns of motion:

$$\dot{\phi}(x, t) = \frac{\delta H}{\delta \pi(x, t)} \quad \dot{\pi}(x, t) = - \frac{\delta H}{\delta \phi(x, t)} \quad (\text{EoM})$$

□ Legendre Transform:

$$H(\phi, \pi) \xrightarrow{\text{LT}} L(\phi, \dot{\phi}) \quad \text{spectator}$$

□ Example: $H := \int \frac{1}{2} \pi(x, t)^2 + V(\phi(x)) d^3x$

$$g(x, t) := \frac{\delta H}{\delta \pi(x, t)}$$

$$\stackrel{\text{EoM}}{=} \dot{\phi}(x, t)$$

Thus:

$$L(\phi, g) = L(\phi, \dot{\phi})$$

$$= \int_{\mathbb{R}^3} \dot{\phi}(x, t) \pi(\phi, \dot{\phi}, x, t) d^3x - H(\phi, \pi(\phi, \dot{\phi}, x, t))$$

← Notice: this is because of the particular π^2 term in H .
On curved space it will be different.

Proposition: The eqns of motion (EoM) are equivalent to:

$$\frac{\delta L}{\delta \phi(x, t)} = \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}(x, t)}$$

Exercise: Check

Euler Lagrange eqn.

Ex ample:

$$H(\phi, \pi) = \int_{\mathbb{R}^3} \frac{\pi^2(x, t)}{2} + \frac{1}{2} \phi(x, t) (m^2 - \Delta) \phi(x, t) d^3x$$

yields: $\dot{\phi}(x, t) = \pi(x, t) \quad \dot{\pi}(x, t) = (-m^2 + \Delta) \phi(x, t)$

i.e: $\ddot{\phi} - \Delta \phi + m^2 \phi = 0$ K.G. eqn.

After Legendre transform:

$$L(\phi, \dot{\phi}) = \int_{\mathbb{R}^3} \frac{\dot{\phi}^2(x, t)}{2} - \frac{1}{2} \phi(x, t) (m^2 - \Delta) \phi(x, t) d^3x$$

yields directly: $-(m^2 - \Delta) \phi = \ddot{\phi}$

Remark: (see arxiv.0810.4293)

- Solving a quantum theory is to do a Fourier transform.
- The lowest order approximation is the Legendre transform.
- The Legendre transform yields the solution to the classical theory.

- Consider the path integral in QFT
(covered in detail later in this course)

$$e^{-iW[J]} = \int e^{iS[\phi]} e^{-i \int J(x) \phi(x) d^4x} \prod_{x \in \mathbb{R}^4} d\phi(x)$$

To know $W[J]$ is to have solved the quantum field theory,
 because it yields all n -point correlation functions $G^{(n)}(x_1, \dots, x_n)$:
 $G^{(n)}(x_1, \dots, x_n) = \frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)}$

Classical action
 Fourier factors (one for each x)
 "Source field"
 (i.e. we integrate "over all fields ϕ " (the proper function space is not known))

$\Rightarrow e^{-iW[J]}$ is the Fourier transform of $e^{iS[\phi]}$.

b) The integrand contributes most, where it is stationary:

$$e^{-iW[J]} \approx e^{iS[\phi] - i \int J \phi d^4x} \left| \text{for that } \phi \text{ for which} \right.$$

$$\left. \frac{\delta}{\delta \phi} (iS[\phi] - i \int J \phi d^4x) = 0 \right.$$

Condition of stationarity of the phase

i.e.

$$W^{(approx)}[J] = \int J \phi d^4x - S'[\phi] \left| \text{where } \phi \text{ obeys} \right.$$

$$\left. \frac{\delta S'}{\delta \phi}(x) = J(x) \right.$$

$$\text{i.e. } W^{(approx)}[J] = \int J \phi[J] d^4x - S'[\phi[J]] \left| \text{where } \phi[J] \right.$$

$$\left. \text{follows from:} \right.$$

$$\left. \frac{\delta S'}{\delta \phi}(x) = J(x) \right.$$

i.e. it's the Legendre transform!

c) So what is knowing $W^{approx}[J]$ good for?

$$\text{Consider } S^{total}[\phi] := S[\phi] - \int J \phi d^4x.$$

As a classical action, it describes a classical field $\phi(x)$ driven by an external "driving force" $J(x)$:

$$\frac{\delta S^{total}}{\delta \phi} = 0, \text{ i.e., } \boxed{\frac{\delta S'}{\delta \phi}(x) = J(x) \quad (EoM)}$$

To solve the classical equations of motion (EoM) is to find the field $\phi(x)$ for any given driving $J(x)$. This is what $W^{approx}[J]$ provides:

$$\boxed{\phi(x) = \frac{\delta W^{approx}[J]}{\delta J(x)}}$$

Because:
(Legendre transform)² = 1