

QFT for Cosmology, Achim Kempf, Lecture 10

Note Title

Recall:

(because the Lagrangian framework treats space and time in the same way)

- * Hamiltonian formulations are suitable for quantization.
- * Lagrangian formulations are suitable to achieve general relativistic covariance.

Strategy:

SR, 1st Q
Hamiltonian formalism

step 1
Legendre transform
(equivalence) →

SR, 1st Q.
Lagrangian formalism

step 2 ↓ allow curvature

GR, 1st Q
Hamiltonian formalism

step 3
Legendre transform
(equivalence) ←

GR, 1st Q
Lagrangian formalism

step 4 ↓

GR, 2nd Q
Hamiltonian formalism

Dyson Schwinger eqns are same
(equivalence) ←

GR, 2nd Q
Lagrangian formalism
(Path integral of QFT)

We already started step 1:

$$\begin{array}{ccc}
 H[\phi, \pi, t] & \begin{array}{c} \xrightarrow{\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} (T)} \\ \xleftarrow{\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} (T^{-1})} \end{array} & L[\phi, \beta, t]
 \end{array}$$

Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:

Lagrangian eqns. of motion:

$$\dot{\phi}(x,t) = \frac{\delta H[\phi, \pi, t]}{\delta \pi(x,t)} \quad (H1)$$

$$\dot{\phi}(x,t) = \beta(x,t) \quad (L1)$$

$$\dot{\pi}(x,t) = - \frac{\delta H[\phi, \pi, t]}{\delta \phi(x,t)} \quad (H2)$$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)} \quad (L2)$$

Proof: We need to show that $(H1 \wedge H2) \xleftrightarrow{T} (L1 \wedge L2)$.

The case " \Rightarrow "

□ Show L1: $\dot{\phi} \stackrel{(H2)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \checkmark$

□ Show L2:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$

$$\stackrel{(H2)}{=} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\stackrel{\text{by def. of } L}{=} - \frac{\delta}{\delta \phi} \left(\int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

$$= - \cancel{\frac{\delta \beta}{\delta \phi} \pi} + \frac{\delta L}{\delta \phi} + \cancel{\frac{\delta L}{\delta \beta} \frac{\delta \beta}{\delta \phi}} \checkmark$$

Exercise: The case " \Leftarrow ".

Result so far:

□ Legendre transform to Lagrangian formulation

\Rightarrow Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a time derivative, no occurrence of space derivatives?) \rightarrow

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)}, \quad \beta(x,t) = \dot{\phi}(x,t)$$

But: How is that advantageous? These equations still seem to treat time differently than space!

Analysis of L_1, L_2 :

We notice: * The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurrences of ϕ in L , including occurrences of $\frac{\partial}{\partial x_i} \phi(x,t)$ in L .

* Why? Because of the definition of $\frac{\delta}{\delta \phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(L[\{\phi(x',t) + \varepsilon \delta^3(x'-x)\}_{x' \in \mathbb{R}^3}] - L[\{\phi(x',t)\}_{x' \in \mathbb{R}^3}] \right)$$

E.g.: $F[u] := \int \sin(x) \left(\frac{d}{dx} u(x) \right) dx$ Is $\frac{\delta F}{\delta u(x)} = 0$? No:

$$= - \int \cos(x) u(x) dx \quad (\text{We assume } u(x) \rightarrow 0 \text{ at boundaries})$$
$$\Rightarrow \frac{\delta F}{\delta u(x)} = -\cos(x)$$

\Rightarrow L_1, L_2 will contain nontrivial time and space derivatives.

* Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial \phi}{\partial x_i}$?

Lemma: Consider any functional Z of the form:

$$Z[f] = \int \text{polynomial} \left(\frac{d}{dx} f \right) dx$$

Then: $\frac{\delta Z}{\delta f(x)} = - \frac{d}{dx} \frac{\delta Z}{\delta \left(\frac{d}{dx} f \right)}$

On the right hand side we view $\frac{d}{dx} f$ as an independent function.

Example:

$$\text{Notation: } \partial_x f(x) = \frac{d}{dx} f(x)$$



$$Z[f] := \int_{\mathbb{R}} (\partial_x f(x))^2 dx'$$

- If we view $\partial_x f$ as an independent function, then we obtain of course:

$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2 \partial_x f(x)$$

- Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -\partial_x \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2 \partial_x \partial_x f(x)$$

- Let us verify this from first principles!

Indeed:

$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\int_{\mathbb{R}} (\partial_{x'} (f(x') + \varepsilon \delta(x-x')))^2 dx' - \int_{\mathbb{R}} (\partial_{x'} f(x'))^2 dx' \right]$$

$$\stackrel{\lim \varepsilon \rightarrow 0}{=} 2 \int_{\mathbb{R}} (\partial_{x'} f(x')) (\partial_{x'} \delta(x-x')) dx'$$

$$\stackrel{\text{int. by parts}}{=} -2 \int_{\mathbb{R}} (\partial_{x'}^2 f(x')) \delta(x-x') dx' + \text{boundary term}$$

$$= -2 \partial_x^2 f(x) \quad \checkmark$$

Recall L2:
$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x, t)} = \frac{d}{dt} \frac{\delta L[\phi, \beta, t]}{\delta \beta(x, t)}$$

Use lemma:

$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x, t)} = \frac{\delta L[\phi, \partial_1 \phi, \partial_2 \phi, \partial_3 \phi, \beta, t]}{\delta \phi(x, t)} - \sum_{j=1}^3 \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_1 \phi, \partial_2 \phi, \partial_3 \phi, \beta, t]}{\delta (\partial_j \phi(x, t))}$$

\Rightarrow L2 takes the form:

$$\frac{\delta L[\phi, \partial_j \phi, \beta, t]}{\delta \phi(x, t)} - \sum_{j=1}^3 \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_j \phi, \beta, t]}{\delta (\partial_j \phi(x, t))} = \frac{d}{dt} \frac{\delta L[\phi, \partial_j \phi, \beta, t]}{\delta \beta(x, t)}$$

Recall also L1: $\beta(x, t) = \dot{\phi}(x, t)$

\rightsquigarrow One is tempted to write:

$$\frac{\delta L[\phi, \partial_j \phi, t]}{\delta \phi(x, t)} \stackrel{?}{=} \sum_{\nu=0}^3 \frac{\partial}{\partial x^\nu} \frac{\delta L[\phi, \partial_\nu \phi, t]}{\delta (\partial_\nu \phi(x, t))} \quad \text{with: } \partial_0 := \frac{d}{dt}$$

However:

Here, we must remember that here the true variable is β , and that we can set $\beta = \dot{\phi}$ only after functional differentiation.

Ramification? □ Can we use the lemma to write

$$\frac{\delta L[\phi, t]}{\delta \phi(x, t)} = 0$$

for the Euler Lagrange field equations? **No!**

□ Because: to apply the lemma to the derivative $\frac{\partial}{\partial t} \phi$, one would need that L possesses a t -integration:

Lemma: For any functional Z of the form:

$$Z[\phi] = \int \text{polynomial} \left(\frac{d}{dx} f \right) dx$$

we have: $\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx} \frac{\delta Z}{\delta \left(\frac{d}{dx} f \right)}$

⇒ The "Action functional":

□ Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi, t] dt$

$S[\phi]$ is called the "action of the field evolution $\phi(x, t)$ "

□ Then, the "Euler Lagrange field equations" are

$$\frac{\delta S[\phi, \partial_r \phi]}{\delta \phi(x, t)} - \sum_{r=0}^3 \frac{\partial}{\partial x^r} \frac{\delta S[\phi, \partial_r \phi]}{\delta (\partial_r \phi)} = 0$$

or equivalently:

$$\frac{\delta S[\phi]}{\delta \phi(x, t)} = 0$$

"The action principle"

□ Notice that the action principle, spelled out, reads:

$$0 = \frac{\delta S[\phi]}{\delta \phi(x,t)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(S[\{\phi(x') + \varepsilon \delta^4(x-x')\}_{x' \in \mathbb{R}^4}] - S[\{\phi(x')\}_{x' \in \mathbb{R}^4}] \right)$$

Example:

The Klein Gordon action:

$$S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} (\partial_0 \phi)^2 - \sum_{j=1}^3 (\partial_j \phi)^2 - m^2 \phi^2 d^4 x$$

□ Using either the action principle or directly the Euler Lagrange field equations, one obtains the Klein Gordon equation (Exercise: verify):

$$\partial_0^2 \phi - \Delta \phi + m^2 \phi = 0, \text{ i.e., } (\square + m^2) \phi(x,t) = 0$$

□ Definitions:

* The action's integrand is called the "Lagrange density" $\mathcal{L}(x,t)$:

$$S[\phi] = \int_{\mathbb{R}^4} \mathcal{L}(x,t) d^4 x$$

* One often formally writes:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \sum_{\mu=0}^3 \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \quad (2)$$

* Notation often used in General Relativity:

a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu} \phi(x,t)$

b.) Twice occurring indices are to be summed over (Einstein summation convention):

E.g., equation (1) can be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

c.) One defines the metric tensor $g_{\mu\nu}(x,t)$.
More about it soon. In special relativity in inertial rectangular coordinate system, we have:

$$g_{\mu\nu}(x,t) = \eta_{\mu\nu} := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

□ Using these definitions, the K.G. action now reads:

$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 d^4x$$

↑ the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same: $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

□ The E.L. eqns read

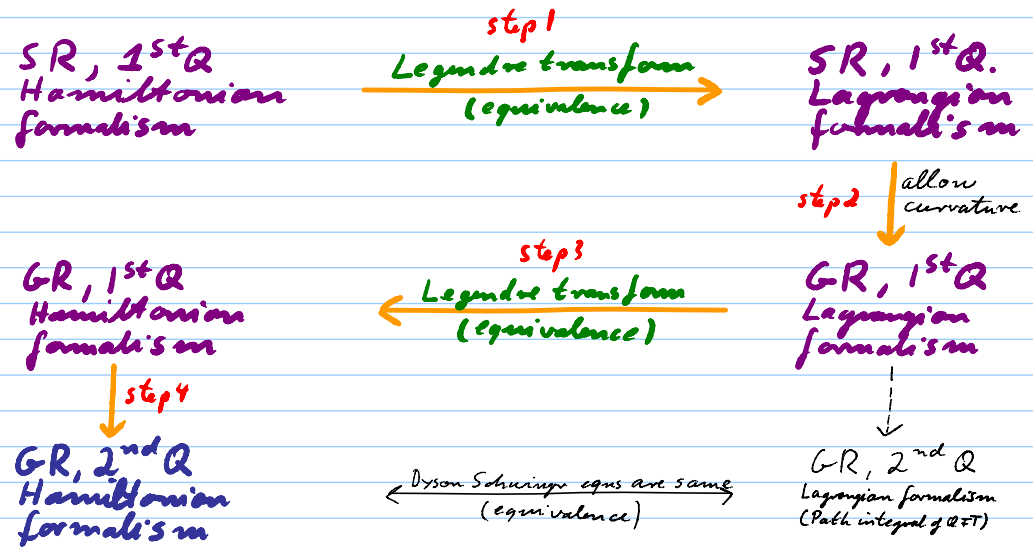
$$\frac{\delta S[\phi, \{\phi_{,\mu}\}]}{\delta \phi(x,t)} = \partial_\mu \frac{\delta S[\phi, \{\phi_{,\mu}\}]}{\delta (\phi_{,\mu}(x,t))}$$

and yield

$$-m^2 \phi = \partial_\mu g^{\mu\nu} \phi_{,\nu}$$

i.e., of course: $(\square + m^2) \phi = 0$

We have now completed Step 1:



Now that we have a beautifully covariant Lagrangian formulation:

Step 2: How to allow for curvature of space-time?

Strategy:

- A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.
- B. Allow arbitrary coordinate systems and allow curvature.

A. Arbitrary coordinate systems

□ Reconsider the K.G. action:

$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 d^4x$$

□ If we change to arbitrary coordinates

$$x^\mu \rightarrow \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then: $\phi(x) \rightarrow \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$

(recall that $\sum_{\nu=0}^3$ is implied)

$$\frac{\partial}{\partial x^\mu} \phi(x) \rightarrow \frac{\partial}{\partial \tilde{x}^\mu} \tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu} \phi(x(\tilde{x})) \right) \frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

□ Therefore, if we transform

$$g^{\mu\nu}(x) \rightarrow \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x(\tilde{x}))$$

then we have that this term in the action

$$g^{\mu\nu}(x) \phi_{,\mu}(x) \phi_{,\nu}(x)$$

is numerically the same in all coordinate systems:

$$\begin{aligned} g^{\mu\nu}(x) \left(\frac{\partial}{\partial x^\mu} \phi(x) \right) \left(\frac{\partial}{\partial x^\nu} \phi(x) \right) &\rightarrow \tilde{g}^{\mu\nu}(\tilde{x}) \left(\frac{\partial}{\partial \tilde{x}^\mu} \tilde{\phi}(\tilde{x}) \right) \left(\frac{\partial}{\partial \tilde{x}^\nu} \tilde{\phi}(\tilde{x}) \right) \\ &= g^{\alpha\beta}(x) \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \left(\frac{\partial}{\partial x^\alpha} \phi(x) \right) \left(\frac{\partial}{\partial x^\beta} \phi(x) \right) \\ &= g^{\mu\nu}(x) \left(\frac{\partial}{\partial x^\mu} \phi(x) \right) \left(\frac{\partial}{\partial x^\nu} \phi(x) \right) \text{ because } \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} = \delta_\nu^\mu \end{aligned}$$



Terminology:

* We say that we let $g^{\mu\nu}(x)$ transform as a contravariant tensor of rank 2.

↑
because upper indices

↑
because 2 indices

* With $g^{\mu\nu}(x) g_{\nu\sigma}(x) = \delta^{\mu}_{\sigma}$ we have

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

Is $S[\phi]$ now coordinate system independent?

No, not yet!

Recall:

As $x^{\mu} \rightarrow \tilde{x}^{\mu}(x)$ the integral measure changes by a Jacobian factor:

$$\int f(x) d^4x \rightarrow \int \underbrace{\tilde{f}(\tilde{x})}_{f(\tilde{x})} \underbrace{\det\left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right)}_{\text{a coordinate-dependent term!}} d^4\tilde{x}$$

A compensating term is needed:

How can we modify the action $S[\phi]$ so that:

- * there is no modification in cartesian coordinates
- * the modification compensates the Jacobian term.

□ Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \right) \overbrace{\sqrt{-\det(g_{\mu\nu})}}^{\uparrow} d^4x$$

□ The volume factor:

* When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓

* Lemma: When $x^\mu \rightarrow \tilde{x}^\mu(x)$ then:

$$\sqrt{|g|} \xrightarrow{\text{short for } \sqrt{-\det(g_{\mu\nu})}} \sqrt{|\tilde{g}|} = \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right) \sqrt{|g|}$$

□ Therefore, we have now in special relativity that the action $S[\phi]$ of a field ϕ comes out the same number, independently of one's choice of coordinate system:

$$\begin{aligned} S[\phi] &\rightarrow \tilde{S}[\tilde{\phi}] = \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|} d^4\tilde{x} \\ &= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right) \det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) \sqrt{|g|} d^4x \\ &= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) \sqrt{|g|} d^4x \\ &= \int \mathcal{L} \det(\delta^{\mu\nu}) \sqrt{|g|} d^4x = \int \mathcal{L} \sqrt{|g|} d^4x \\ &= S[\phi] \end{aligned}$$

B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \\ & & & -1 \end{pmatrix}$

can look very nontrivial in generic

coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

→ These metrics belong to spaces with curvature. We need not change the action's formula: Just allow arbitrary metrics $g_{\mu\nu}(x)$.

□ we saw that in generic (i.e. arbitrarily chosen) coordinates $\tilde{x}^r = \tilde{x}^r(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \quad (c)$$

⇒ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

Key Question:

Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

by changing coordinates according to $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta}$?

Answer: **No!** The others describe "curved" spacetimes.

A given spacetime can be described by any one of an equivalence class $[g]$ of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

Definition: Each equivalence class $[g]$ is called a **Riemannian or Lorentzian Structure**, depending on the signature of the metric.

How many Lorentzian or Riemannian structures are there?

Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: In n dimensions, the metric g has n^2 component functions $g_{\mu\nu}(x)$.

Because of $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, only $n(n+1)/2$ are independent.

But we can choose n functions $\tilde{x}^\alpha(x)$ in $\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} g_{\alpha\beta}$.

A: $D = \underbrace{n(n+1)/2}_{\text{\# of indep elements of a symmetric } n \times n \text{ matrix } g_{\mu\nu}} - \underbrace{n}_{\text{\# of change of coordinate functions } \tilde{x}^\alpha = \tilde{x}^\alpha(x)}$

Examples: For $n=1+3$, have $D=6$. For $n=2$, have $D=1$.