QFT for Cosmology, Achim Kempf, Lecture 11

Recall the strategy:

$$
\begin{array}{ccc}
\text{SR, } 1^{st}\text{Q Hamiltonian formalism} & \xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}} & \text{SR, } 1^{st}\text{Q. Lagrangian formalism} \\
 & & \Big\downarrow \text{step 2: allow curvature} \\
\text{GR, } 1^{st}\text{Q Hamiltonian formalism} & \xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}} & \text{GR, } 1^{st}\text{Q Lagrangian formalism} \\
\Big\downarrow \text{step 4} & & \Big\downarrow \\
\text{GR, } 2^{nd}\text{Q Hamiltonian formalism} & \xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}} & \text{GR, } 2^{nd}\text{Q Lagrangian formalism (Path integral of QFT)}
\end{array}
$$

we are here

Curvature:

- We postulate the coordinate system-independent Klein Gordon action:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi, x_i)\right)\sqrt{|g|}\; d^4x$$

- We will allow almost arbitrary metric tensors $g_{\mu\nu}(x)$, even those for which there do not exist coordinates $\tilde{x}$ in which:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x$$

- But we must have that, at least locally, special relativity holds!

$\Longrightarrow$ Consider only $g_{\mu\nu}(x)$ for which for each x_0, there exists a change of coordinates

$$x \to \tilde{x}$$

so that: $$\tilde{g}_{\mu\nu}(\tilde{x}_0) = \eta_{\mu\nu}$$

- This requirement is **The Equivalence Principle**:

  * We postulate that gravity can always locally be eliminated:

  * We assume that if a freely falling observer in a small region sets up a rectangular coordinate system the observer will see arbitrarily small gravity effects if the region is made arbitrarily small.

  * For this to be true, any body's gravitational mass must be equal to its inertial mass, i.e. all bodies must fall equally. (Else the notion of freely falling observer) is not even well defined)

How can one identify the presence of curvature?

* Assume we are given a metric tensor

$$g_{\mu\nu}(x)$$

as an explicit matrix-valued function, in some coordinates.

* How can we determine whether or not this is, e.g., the metric tensor of flat space-time, i.e., whether or not there exist coordinates $\tilde{x}$ in which $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$?

* This problem is solved in differential geometry:

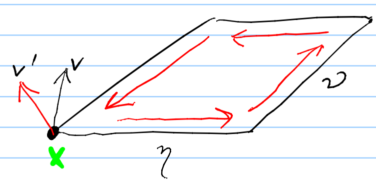
Define: The "Christoffel symbol functions":

$$\Gamma^{\mu}_{\alpha\beta}(x) := \frac{1}{2} g^{\mu\nu}(x)\Big(g_{\alpha\nu,\beta}(x) + g_{\beta\nu,\alpha}(x) - g_{\alpha\beta,\nu}(x)\Big)$$

Define: The "Riemann Curvature Tensor":

$$R^i{}_{jkl}(x) := \Gamma^i{}_{lj,k}(x) - \Gamma^i{}_{kj,l}(x) + \Gamma^s{}_{lj}(x)\,\Gamma^i{}_{ks}(x) - \Gamma^s{}_{kj}(x)\,\Gamma^i{}_{ls}(x)$$

It's rôle? A space is called curved at x if the parallel transport of a vector v along an infinitesimal parallelogram returns the vector v' to x, but v' is rotated by some amount. $R^\mu{}_{\nu\rho\beta}$ tells by how much:



$$\boxed{(v'-v)^a = \eta^b \nu^c R^a{}_{bcd}(x)\, v^d}$$

Remark:

If the parallelogram does not even close we say that space-time has "Torsion". There is no evidence for torsion in nature.

Proposition:

Assume that, in a region, A, of space-time:

$$R^\mu{}_{\nu\rho\varepsilon}(x) = 0 \quad \text{for all } x \in A$$

Then and only then there exist coordinates $\tilde{x}$ so that:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x \in A$$

The dynamics of space-time

Problem: What are the equations of motion for spacetime's curvature?

Which are the degrees of freedom of curvature for which we have to find an equation of motion?

- We saw that the curvature of space-time is encoded in the matrix-valued metric function:

$$g_{\mu\nu}(x)$$

- However, if $g_{\mu\nu}(x)$ looks nontrivial, this can be for two different reasons:

1. Spacetime has little or no curvature and $g_{\mu\nu}(x)$ is nontrivial just because of an unlucky choice of coordinates.
2. Spacetime is curved, i.e., we can not make $g_{\mu\nu}(x)$ take the form $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$ for all $\tilde{x}$ no matter which coordinates we choose.

- Therefore, it is difficult to pinpoint in the matrix function $g_{\mu\nu}(x)$ the curvature degrees of freedom.
- And: Even the entries $R^{\mu}{}_{\nu\rho\varepsilon}$ of the curvature tensor are coordinate system dependent.

Strategy:

- Use the degrees of freedom of curvature to build a scalar and therefore coordinate system independent function $S_{grav}[g]$
- Then, use this function as the action for gravity.
- The equations of motion for gravity should follow from the action principle (and they do).

⇝ Need to define a scalar that encodes curvature!
We begin by going from a 4-tensor to a 2-tensor:

Definition: The "Ricci tensor":

$$\boxed{R_{\mu\nu}(x) := R^{i}{}_{\mu i \nu}(x)}$$

↙ Recall: $\sum_{i=0}^{3}$ is implied

Note: Other index contractions would vanish because of antisymmetries of $R^{r}{}_{\nu s \varepsilon}(x)$ that are implied by the definition of $R^{r}{}_{\nu s \varepsilon}(x)$.

Remark:

$R_{\mu\nu}(x)$ carries strictly less information than the full Riemann curvature tensor:

- $*$ If $R_{\mu\nu}(x)=0$ it is still possible that $R^{r}{}_{\nu s \varepsilon}(x) \neq 0$!
- $*$ This happens to be the case, e.g., for gravitational waves.

Definition: The "curvature scalar" (or "Ricci scalar")

$$R(x) := g^{\mu\nu}(x)\, R_{\mu\nu}(x)$$

Other curvature scalars:

* The simplest scalar that can be formed from the metric alone is $g^{\mu\nu}(x)\, g_{\nu\mu}(x) = 4$.
* The next simplest scalar that can be formed is the Ricci scalar $R(x)$.
* All other scalars made out of g only are composed of higher powers of the Riemann tensor $R^{\mu}{}_{\nu\sigma\varepsilon}(x)$:

$$R_{\mu\nu} R^{\mu\nu}, \quad R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} \quad \text{etc.}$$

The gravitational action

- A priori, the full action now reads:

other "matter" fields for e^-, quarks, photons etc

$$S'_{tot}[g, \phi, e_i] = S_{KG} + S_{other} + S'_{grav}$$

Klein Gordon action

with: $$S'_{grav}[g] := \int \left(c_0 + c_1 R(x) + c_2 R_{\mu\nu}(x) R^{\mu\nu}(x) + c_3 R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} + \dots \right) \sqrt{|g|}\, d^4x$$

- Comparison with experiment shows evidence only for the first two terms:

$$S_{grav}[g] = -\frac{1}{16\pi G} \int \left(2\Lambda + R(x) \right) \sqrt{|g(x)|}\, d^4x$$

"Einstein action" Newton's constant "Cosmological constant"

Remark: D. Lovelock here determined all generalizations to higher terms and higher dimensions that still possess 2^{nd} order initial value problems.

at Applied Math at UW

The equations of motion

The action principle is to require that the action be extremal with respect to all degrees of freedom:

A) $\frac{\delta S'_{tot}}{\delta \ell_i(x)} = 0$ B) $\frac{\delta S'_{tot}}{\delta g_{\mu\nu}(x)} = 0$ C) $\frac{\delta S'_{tot}}{\delta \phi(x)} = 0$

A) Require: $$\frac{\delta S'_{tot}[g, \ell, \phi]}{\delta \ell_i(x)} = 0$$

This yields the general relativistically covariant field equations for all "other" fields. (We will ignore the $\ell_i(x)$ for now.)

Quantization: Legendre transform $\to H(\ell_i, \pi_{\ell_i}) \to$ impose $[\ell_i, \pi_{\ell_i}] = i\delta$ etc.

B) Require: $$\frac{\delta}{\delta g_{\mu\nu}(x)} S'_{tot}[\phi, \ell_i, g] = 0$$

This yields the equation of motion for the dynamics of curvature, i.e., the Einstein equation:

(See exercise in Mukhanov's text) $\rightarrow$

$$\boxed{\underbrace{R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x)}_{\sim \frac{\delta S'_{grav}}{\delta g_{\mu\nu}(x)}} = + 8\pi G \underbrace{T_{\mu\nu}(x)}_{\sim -\frac{\delta (S'_{other} + S'_{KG})}{\delta g_{\mu\nu}(x)}}}$$

* Here, $T_{\mu\nu}(x)$ is the "Energy Momentum Tensor".
Neglecting the contribution by the $\ell_i(x)$, one obtains:

$$T^{(K.G.)}_{\mu\nu}(x) = \phi_{,\mu}(x)\, \phi_{,\nu}(x) - g_{\mu\nu}(x) \left(\frac{1}{2} g^{\alpha\beta}(x)\, \phi_{,\alpha}(x)\, \phi_{,\beta}(x) - \overbrace{V(\phi(x))}^{\text{mass term } m^2\phi^2 \text{ included}} \right)$$

* **Quantization:** To quantize the Einstein equation is difficult for many reasons:

  - For example, it is difficult to separate the curvature degrees of freedom from mere artifacts of the choice of the coordinate system.

  - Also, the Einstein equation is highly nonlinear.

  - So far, all attempts have run into severe difficulties, even perturbative approaches.

⇝ **This course:** 1.) We will have initially consider known classical solutions $g_{\mu\nu}(x)$ and quantize only $\phi(x)$.

2.) Then, we will quantize linear perturbations of the metric.

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c) Require: $$\frac{\delta}{\delta\phi(x)} S_{tot}[g,\phi] = 0$$

- Since ϕ occurs only in S_{KG} we have, equivalently:

$$\frac{\delta S_{KG}}{\delta\phi(x)} = 0$$

- Recall S_{KG}:

$$S_{KG}[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \Big(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \overbrace{\lambda\phi^4}^{\text{Example of a potential}} \Big) \sqrt{|g|}\, d^4x$$

- Apply the Euler Lagrange equations:

$$\frac{\delta S_{KG}[\phi,\{\phi_{,\mu}\},g]}{\delta\phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\},g]}{\delta(\phi_{,\mu}(x,t))}$$

$\square$ $\Rightarrow$ Klein Gordon equation in general relativity:

$$\left(-\frac{1}{2}m^2 2\phi(x) - \frac{1}{2}\lambda 4\phi^3(x)\right)\sqrt{|g(x)|} = \partial_\mu\left(\frac{2}{2}g^{\mu\nu}(x)\,\phi_{,\nu}(x)\sqrt{|g(x)|}\right)$$

Thus: $$\boxed{\frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(x)\sqrt{|g(x)|}\,\phi_{,\nu}(x)\right) + m^2\phi(x) + 2\lambda\phi^3(x) = 0}$$

$\square$ Definition: The "d'Alembert operator", $\square$

$$\square := \frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^\mu}\,g^{\mu\nu}(x)\sqrt{|g(x)|}\frac{\partial}{\partial x^\nu}$$

Thus: $$\boxed{\square\,\phi(x) + m^2\phi(x) + 2\lambda\phi^3(x) = 0}$$

Next: Step 3 in

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{SR, } 1^{st}Q \\ \text{Hamiltonian formalism}\end{array} \arrow[r, "\substack{\text{step 1} \\ \text{Legendre transform}}", "\text{(equivalence)}"'] & \begin{array}{c}\text{SR, } 1^{st}Q. \\ \text{Lagrangian formalism}\end{array} \arrow[d, "\text{allow curvature}", "\text{step 2}"'] \\
\begin{array}{c}\text{GR, } 1^{st}Q \\ \text{Hamiltonian formalism}\end{array} \arrow[d, "\text{step 4}"] & \begin{array}{c}\text{GR, } 1^{st}Q \\ \text{Lagrangian formalism}\end{array} \arrow[l, "\substack{\text{step 3} \\ \text{Legendre transform}}"', "\text{(equivalence)}"] \arrow[d, dashed, "\text{(PI)}"] \\
\begin{array}{c}\text{GR, } 2^{nd}Q \\ \text{Hamiltonian formalism}\end{array} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \begin{array}{c}\text{GR, } 2^{nd}Q \\ \text{Lagrangian formalism} \\ \text{(Path integral of QFT)}\end{array}
\end{tikzcd}$$

Comment on step (PI): 2^{nd} quantization with path integral

- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system.

- Then, for each field $\phi(\vec{x},t)$ we can calculate its action $S_{KG}[\phi,g]$:

$$S_{KG}[\phi,g] = \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} - m^2\phi^2 - \lambda\,\phi^4\right)\sqrt{|g|}\;d^4x$$

- Following Feynman, we obtain "probability amplitudes":

$$\text{prob. ampl.}\,[\phi] := e^{\frac{i}{\hbar}S_{KG}[\phi,g]}$$

- Consider, e.g., the vacuum expectation value of $\phi(\vec{x},t)\,\phi(\vec{x}',t)$, i.e., the correlation function of field amplitudes:

$$G(\vec{x},t,\vec{x}',t) := \langle 0|\,\hat{\phi}(\vec{x},t)\,\hat{\phi}(\vec{x}',t)\,|0\rangle$$

- We will later see how to calculate it using commutation relations etc.

- With Feynman we also get it from the path integral:

$$G(\vec{x},t,\vec{x}',t) = N\int_{\text{all }\phi}\phi(\vec{x},t)\,\phi(\vec{x}',t)\;e^{\frac{i}{\hbar}S_{KG}[\phi,g]}\,D[\phi]$$

↑ "Path Integral" over a function space

Advantages:

1) Quick derivation of Feynman rules

2) Manifestly covariant.

Problems:

1) Ill defined due to uncountable number of integrations and related divergencies.

2) Even when these are temporarily regularized, the identification of the vacuum state is ambiguous.
→ This issue is better handled in canonical formalism.

Recall strategy:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{SR, } 1^{st}\text{Q Hamiltonian formalism} \arrow[r, "\text{step 1: Legendre transform}", "\text{(equivalence)}"'] & \text{SR, } 1^{st}\text{Q. Lagrangian formalism} \arrow[d, "\text{allow curvature}", "\text{step 2}"'] & \\
\text{GR, } 1^{st}\text{Q Hamiltonian formalism} \arrow[d, "\text{step 4}"] & \text{GR, } 1^{st}\text{Q Lagrangian formalism} \arrow[l, "\text{step 3: Legendre transform}"', "\text{(equivalence)}"] \arrow[d, dashed] & \text{we are here} \\
\text{GR, } 2^{nd}\text{Q Hamiltonian formalism} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \text{GR, } 2^{nd}\text{Q Lagrangian formalism (Path integral of QFT)} &
\end{tikzcd}
$$

Recall:

General relativistic covariant Klein Gordon theory in the Lagrangian formulation (neglecting the potential):

$$S_{KG} = \frac{1}{2} \int_{\mathbb{R}^4} \left(g^{\mu\nu}(x)\, \phi_{,\mu}(x)\, \phi_{,\nu}(x) - m^2 \phi^2(x) \right) \sqrt{|g|}\, d^4x$$

↖ we assume that the coordinate system is such, for simplicity.

It yields, via $\frac{\delta S_{KG}}{\delta \phi} = 0$ the Klein Gordon eqn:

$$\frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x^\mu} \left(g^{\mu\nu}(x) \sqrt{|g(x)|}\, \phi_{,\nu}(x) \right) + m^2 \phi(x) = 0 \qquad \text{(KG)}$$

* We read off the Lagrangian:

$$L_{KG}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}(x,t)\,\phi_{,\mu}(x,t)\,\phi_{,\nu}(x,t) - m^2\phi^2(x,t)\right)\sqrt{|g|}\,d^3x$$

Step 3: Legendre transform back to the Hamiltonian form

* The transform:

$$H(\phi,\pi,t) \xleftarrow[\text{Legendre transform}]{\pi(x,t) := \frac{\delta L}{\delta \phi_{,0}(x,t)}} L(\phi,\phi_{,0},t)$$

* Thus, the canonically conjugate field $\pi(x,t)$ reads:

$$\pi(x,t) = \frac{\delta L}{\delta\phi_{,0}(x,t)} = \sqrt{|g(x,t)|}\; g^{0\mu}(x,t)\,\phi_{,\mu}(x,t)$$

* Explicitly:

$$\pi(x,t) = \sqrt{|g|}\,g^{00}\phi_{,0} + \sum_{i=1}^{3}\sqrt{|g|}\,g^{0i}\phi_{,i}$$

* Thus, we can also express $\phi_{,0}(x,t)$ in terms of $\phi(x,t)$ and $\pi(x,t)$ (as will be necessary after the Legendre transform):

(of course, $g^{\mu\nu}$ depends on x,t too)

$$\phi_{,0}(x,t) = \frac{\pi(x,t)}{g^{00}\sqrt{|g|}} - \sum_{i=1}^{3}\frac{g^{0i}}{g^{00}}\,\phi_{,i}(x,t) \qquad \text{(V)}$$

* The Hamiltonian:

$$H(\phi,\pi) = \int_{\mathbb{R}^3}\pi(x,t)\,\phi_{,0}(x,t)\,d^3x - \underbrace{\frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\sqrt{|g|}\,d^3x}_{=L(\phi,\phi_{,0}(\phi,\pi),t)}$$

Why don't we need a factor of $\sqrt{|g|}$ for covariance here? Because π has it built in!

* In H, one needs to express all occurring $\phi_{,0}$ in terms of the new variables ϕ and π, by using (V), to obtain $H(\phi,\pi,t)$.

→ **Exercise:** Calculate $H(\phi,\pi,t)$ and simplify the expression as far as possible.

* **The equations of motion:**

We know from the general properties of the Legendre transform that the equations of motion now take the form:

$$\frac{d}{dt}\phi(x,t) = \frac{\delta H(\phi,\pi,t)}{\delta \pi(x,t)}\ , \quad \frac{d}{dt}\pi(x,t) = -\frac{\delta H(\phi,\pi,t)}{\delta \phi(x,t)}$$

* **Exercise:** Verify that these eqns are equivalent to (KG).

We are now ready to 2^{nd} quantize: