Note Title

QFT for Cosmology, Achim Kempf, Lecture 12

We are now ready to 2nd quantize:

SR, 1st Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, 1st Q. Lagrangian formalism

step 2 — allow curvature

GR, 1st Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1st Q Lagrangian formalism

step 4

GR, 2nd Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, 2nd Q Lagrangian formalism (Path integral of QFT)

Solving the quantized theory is to solve:

1.) Commutation relations:

$$\left.\begin{array}{l} [\hat\phi(x,t), \hat\pi(x',t)] = i\hbar\,\delta^3(x-x') \\ [\hat\phi(x,t), \hat\phi(x',t)] = 0 \\ [\hat\pi(x,t), \hat\pi(x',t)] = 0 \end{array}\right\} \quad \text{(CCRs)}$$

2.) Hermiticity:

$$\hat\phi^+(x,t) = \hat\phi(x,t), \qquad \hat\pi^+(x,t) = \hat\pi(x,t) \qquad \text{(HC)}$$

which is needed so that the expectation values are real.

3) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat{f}(\phi,\pi) = \frac{1}{i\hbar}[\hat{f}, \hat{H}] \qquad \text{for } \hat{f} = \hat{\phi},\ \hat{f} = \hat{\pi},\ \text{etc}$$

Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad \text{(EoM2)}$$

How to solve the CCR, HC and EoM equations?

The a_k, a_k^+ take care of the CCRs

Recall: the solution we obtained on Minkowski space:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(\underbrace{e^{-i\omega_k t + ikx}}\, a_k + \underbrace{e^{i\omega_k t - ikx}}\, a_k^+\right) d^3k$$

Number-valued solutions to the K.G. equation

and $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$

Strategy:

- ensure hermiticity, HC, by construction
- separate the CCR and EoM problems:

Ansatz:

$$\boxed{\begin{aligned} \hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \end{aligned}}$$

(k need not be a "momentum"!)

- Here, we use the easy-to-construct operators that obey

$$[a_k, a^+_{k'}] = \delta_{k,k'}$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

- Try out the ansatz:

  * Hermiticity: ✓

  (HC) holds by construction.

* The 1^{st} equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu}+m^2\right)\hat{\phi}(x,t)=0 \qquad (\text{EoM1})$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\,a_k + u_k^*(x,t)\,a_k^+$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2^{nd} equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

Checking the CCRs:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] \overset{!}{=} i\hbar\, \delta^3(x-x')$$

- Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

- Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

$$\Rightarrow \quad \hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu} \sum_k \left[\left(\frac{\partial}{\partial x^\nu} u_k(x,t) \right) a_k + \left(\frac{\partial}{\partial x^\nu} u_k^*(x,t) \right) a_k^+ \right]$$

Now check CCR:

$$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$$

$$= \left[\sum_k u_k(x',t)\, a_k + u_k^*(x',t)\, a_k^+ ,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left(\left(\frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) \right) a_{k'}^+ \right) \right]$$

$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left(u_k(x',t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x',t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_k(x,t) \right) \overset{!}{=} i\, \delta^3(\vec{x} - \vec{x}')$$

Conclusion so far:

Our ansatz

$$\hat{\phi}(x,t) := \sum u_n(x,t)\, a_n + u_n^*(x,t)\, a_n^+$$

solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

(W) $$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^{\nu}} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^{\nu}} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

When do such $\{u_k(x,t)\}$ exist? I.e., when does the ansatz succeed?

Proposition:
- Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form:

  $\mathbb{R} \times \mathcal{M}$

  ↑ any 3-dim differentiable manifold

- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
- Then, such a set of functions $\{u_k\}$ can be shown to exist.
- In fact there are many such sets $\{\tilde{u}_k\}$ obeying (W)!

  (And we will have to address which set to choose to solve the theory.)

Proof:

- Consider the vector space, V, of all real-valued solutions of the Klein Gordon equation.

- We define a bi-linear form $(\, , \,)$ on V. For all $f, h \in V$:

$$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f)$$

Σ ← any spacelike hypersurface i.e. set of points of equal time.

- Proposition: (f,h) is independent of choice of Σ.
  Proof: Later (uses Stokes' theorem and K.G. equation)

- (f,h) is a symplectic form, i.e.: $(f,h) \underset{\text{easy to see}}{=} -(h,f)$.

- What can we do with $(\, , \,)$? No diagonalization?

  Theorem (Darboux):

  For any nondegenerate symplectic form $(,)$, there exists a basis $\{v_n\}$ such that, in this basis, $(,)$ takes the matrix form:

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ 0 & & & \ddots \end{pmatrix}$$

  i.e., such that $(v_{2n}, v_{2n+1}) = 1$, $(v_{2n+1}, v_{2n}) = -1$ and all other pairings vanish.

- Thus, if we expand $a, b \in V$ as: $\underset{\in V}{a} = \underset{\in \mathbb{R}}{a_n} \underset{\in V}{v_n}$, $b = \underset{\in \mathbb{R}}{b_n} v_n$

  Then: $(a,b) = \sum_{n=0}^{\infty} a_{2n} b_{2n+1} - a_{2n+1} b_{2n}$

- Now assume we picked such a basis $\{v_n\}$ in V.
- Recall: V = space of real-valued solutions of K.G. eqn.
- Definition:

$\overline{V}$:= space of complex-valued solutions of K.G. eqn.

- We easily find a basis of $\overline{V}$, namely $\{u_n\} \cup \{u_n^*\}$ where:

$$u_n := \frac{1}{\sqrt{2}}\left(v_{2n} + i\, v_{2n+1}\right), \quad u_n^* = \frac{1}{\sqrt{2}}\left(v_{2n} - i\, v_{2n+1}\right)$$

- What is a natural product $\langle \, , \, \rangle$ on $\overline{V}$?

- On $\overline{V}$ we define:

$$\langle f, h \rangle := i \int_{\Sigma} d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} \left(f^* \partial_\nu h - (\partial_\nu f^*) h \right)$$

- Then, $(\, , \,)$ yields:

$$\langle u_n, u_m \rangle = -\delta_{n,m}, \quad \langle u_n^*, u_m^* \rangle = +\delta_{n,m}, \quad \langle u_n, u_m^* \rangle = 0 \qquad (I)$$

Exercise: verify this.

Thus, $\langle \, , \, \rangle$ is an indefinite inner product on $\overline{V}$: $\langle \, , \, \rangle = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Proposition: A resolution of the identity on $\overline{V}$ is given by:

$$1\!\!1 = \sum_n -|u_n\rangle\langle u_n| + |u_n^*\rangle\langle u_n^*|$$

Remark: One can also turn $\bar{V}$ into a Hilbert space, namely the Krein space: Let P^+ and P^- be the projectors on the spaces spanned by the u_k^* and the u_k respectively.
Then, $\langle\langle f,g\rangle\rangle := \langle f, P^+ g\rangle - \langle f, P^- g\rangle$ is a positive definite inner product, and the Krein space $(\bar{V}, \langle\langle , \rangle\rangle)$ is a Hilbert space.

Proof: Indeed, $\mathbb{1}|u_n\rangle = |u_n\rangle$ using $\langle u_n, u_n\rangle = -1$

$\mathbb{1}|u_n^*\rangle = |u_n^*\rangle$ using $\langle u_n^*, u_n^*\rangle = 1$.

so that for any $|f\rangle \in \bar{V}$ we have:

$$-\sum_n |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle \qquad \text{(P)}$$

Writing this out, we will now show that it yields (W), i.e.:

$$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

Indeed, using

$$\langle f, h\rangle = i \int_\Sigma d\Sigma_\mu \sqrt{|g|}\, g^{\mu\nu} \left(f^* \partial_\nu h - (\partial_\nu f^*) h \right)$$

which reads, in a suitable coordinate system:

$$= i \int d^3x' \sqrt{|g(x')|}\, g^{0\nu}(x') \left(f^*(x') \partial_\nu h(x') - \partial_\nu f^*(x') h(x') \right)$$

we see from (P) that $\forall f \in \bar{V}$:

$$-\sum_n |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle$$

reads:

$$\sum_n u_n(x,t)\, i\int_\Sigma d^3x' \sqrt{g}\, g^{0\nu}\left(u_n^* \partial_\nu f - (\partial_\nu u_n^*) f\right)$$

$$-\sum_n u_n^*(x,t)\, i\int_\Sigma d^3x' \sqrt{g}\, g^{0\nu}\left(u_n \partial_\nu f - (\partial_\nu u_n) f\right) = f(x,t)$$

Now, interchanging $\sum_n$ and $\int_\Sigma$ yields $\forall f \in \bar{V}$:

$$i\int_\Sigma d^3x \sqrt{|g(x')|}\, g^{0\nu}(x') \sum_n \Big(u_n(x,t)\, u_n^*(x',t_0)\,\partial_{x',\nu} - u_n(x,t)\,\partial_{x',\nu} u_n^*(x',t_0)$$

$$- u_n^*(x,t)\, u_n(x',t_0)\,\partial_{x',\nu} + u_n^*(x,t)\,\partial_{x',\nu} u_n(x',t_0)\Big) f(x',t_0) = f(x,t) \qquad (*)$$

Notice:

On the left hand side of this equation, f is evaluated for all x' but only at the one time, say t_0, of Σ.

- Now choose an arbitrary function $g(x')$.
- Then there exists a solution $f(x',t)$ of the Klein Gordon equation obeying:

1) $f(x',t_0) = g(x')$
2) $g^{0\nu}(x',t_0)\,\partial_{x',\nu} f(x',t_0) = 0$

(Because the 2nd order K.G. equation on a globally hyperbolic spacetime has a well-defined Cauchy problem)

- Therefore, $(*)$ yields, for all choices of $g(x)$:

$$i\int_\Sigma d^3x' \sqrt{|g(x')|}\, g^{0\nu}(x') \sum_n \Big(- u_n(x,t)\,\partial_{x',\nu} u_n^*(x',t) + u_n^*(x,t)\,\partial_{x',\nu} u_n(x',t)\Big) g(x') = g(x) \quad \forall g(x)$$

$$\Rightarrow \boxed{\sqrt{|g|}\, g^{0\nu}(x) \sum_k \left(u_k(x,t)\frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t)\frac{\partial}{\partial x'^\nu} u_k(x',t)\right) = i\,\delta^3(x-x')} \quad (W) \ ✓$$

Conclusion so far:

1.) Commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\, \delta^3(x-x')$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0, \quad [\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$$

2.) Hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$, $\quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t)$

3) Equations of motion:

$$\left(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu} + m^2\right) \hat{\phi}(x,t) = 0 \qquad (KG)$$

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

can always be solved with this ansatz on any globally hyperbolic spacetime:

$$\boxed{\begin{aligned} \hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \end{aligned}}$$

where $[a_k, a_{k'}^+] = \delta(k-k')$ and where the $u_k(x,t)$ are number-valued solutions to (KG) which also obey (R1):

$$\boxed{\sqrt{|g|}\, g^{0\nu}_{(x')} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

We proved that such $u_k(x,t)$ always can be found.

Outlook:

Q: We showed that (W) ensures the CCRs for one time foliation $\mathbb{R} \times \mathcal{M}$.
What guarantees the CCRs for all foliations?

A: Stokes' theorem and unitarity.

Q: Are the $u_k(x,t)$ unique?

A: No! Math: Bogolubov transformations
$\uparrow$

Q: What's the physics?

A: Vacuum ambiguity.