Note Title

Recall: The free Klein Gordon quantum field in a generic curved space-time must obey:

$$\hat{\phi}^+(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^+(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

$$i\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}(t)], \quad i\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}(t)] \qquad \text{(EoM)}$$

which can be written in this form:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0, \quad \hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad \text{(EoM)}$$

And: On all spacelike hypersurfaces, Σ, the CCRs must hold:

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0, \quad [\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0, \quad [\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x') \qquad \text{(CCR)}$$

We want to show: The following ansatz for $\hat{\phi}(x,t)$ succeeds:

$$\hat{\phi}(x,t) := \sum_k \underbrace{u_k(x,t)}_{\text{number-valued solutions to K.G. eqn.}} a_k + u_k^*(x,t)\, a_k^+, \quad \text{with } [a_k, a_{k'}^+] = \delta_{k,k'}$$

at least if the spacetime is globally hyperbolic.

So far we showed:

- The HC and EoM obeyed at all time.
- In a fixed coordinate system, CCRs are obeyed $\forall t$ if $\{u_k\}$ obey $\forall t$:

$$\boxed{\sqrt{|g|}\, g^{0\nu}\sum_k \left(u_k(x,t)\frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t)\frac{\partial}{\partial x'^\nu} u_k(x',t)\right) = i\,\delta^3(x-x')} \qquad \text{(W)}$$

- Using Darboux's theorem, we showed that there exists a set of solutions $\{u_k\}$ so that (W) holds at some time t_0

Conservation of the CCRs? This is implied by the self-adjointness of $\hat{H}$:

- As always in quantum theory, the time evolution operator $\hat{U}(t,t_0) = T e^{i\int_{t_0}^{t} \hat{H}(t')dt'}$ is unitary: $\hat{U}^+ = \hat{U}^{-1}$.
- It allows one to express the time evolution of the observables, such as field operators, through:

$$\hat{\phi}(x,t) = \hat{U}(t,t_0)\,\hat{\phi}(x,t_0)\,\hat{U}^+(t,t_0)$$

$$\hat{\pi}(x,t) = \hat{U}(t,t_0)\,\hat{\pi}(x,t_0)\,\hat{U}^+(t,t_0)$$

- Thus: $[\hat{\phi}(x,t),\hat{\pi}(x',t)] = [\hat{U}\hat{\phi}(x,t_0)\hat{U}^{-1}, \hat{U}\hat{\pi}(x',t_0)\hat{U}^{-1}]$

$$= \hat{U}\,[\hat{\phi}(x,t_0),\hat{\pi}(x',t_0)]\,\hat{U}^{-1}$$

$$= \hat{U}\, i\,\delta^3(x-x')\,\hat{U}^{-1} = i\,\delta^3(x-x')$$

Problem: Is the quantization coordinate system independent?

Assume we solve the theory as above.

Now if we change coordinate system, and therefore the choices of $\{\Sigma\}$, would the CCRs still hold on every spacelike hypersurface Σ'?

Proposition: Yes: if CCRs hold in one coordinate system, then they hold in all: The CCRs keep holding when deforming a Σ to a Σ'.

Proof: Rewrite the symplectic form (f,h) more abstractly:

Recall: $f,g \in V$ are solutions of KG eqn.

$$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu}(f\,\partial_\nu h - h\,\partial_\nu f)$$

a differential 3-form (Recall: Only 3-forms have 3-dim integrals)

$$= \int_\Sigma \tilde{j}$$

Here, we defined the contravariant vector field

$$j^\mu(x,t) := g^{\mu\nu}\left(f\,\partial_\nu h - h\,\partial_\nu f\right)$$

and from it the differential 3-form:

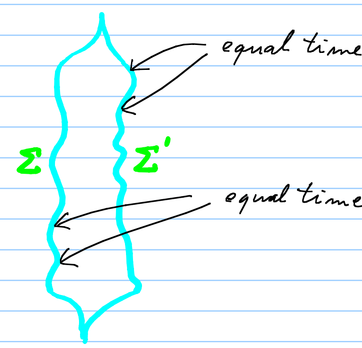
$$\tilde{j} := i_j\,\Omega$$

inner derivation

3-form

Volume 4-form $\sqrt{|g|}\,d^4x$

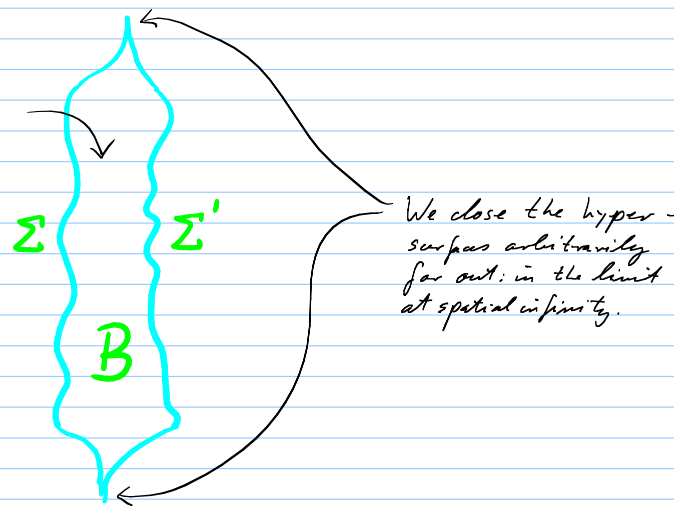
We need to show that the value of the symplectic form stays the same when deforming Σ:



- Now integrate over both Σ and Σ':

Σ and Σ' enclose the 4-dim. volume B

$$\Sigma \cup \Sigma' = \partial B$$



- Use Stokes' theorem:

$$\int_{\partial B} \tilde{j} = \int_B d\tilde{j}$$

Notice:

If we can show $d\tilde{j}=0$ we are done!

That's because then:

$$0=\int_{\Sigma\cup\Sigma'}\tilde{j}=\int_{\Sigma}\tilde{j}+\int_{\Sigma'}\tilde{j}=-\int_{\Sigma}\tilde{j}+\int_{\Sigma'}\tilde{j}$$

Both j pointing out of B, i.e. one to the future one to the past.

Both j future pointing.

$\Rightarrow \int_{\Sigma}\tilde{j}$ is indeed indep. of choice of Σ, if we can show $d\tilde{j}=0$.

Indeed:

$$d\tilde{j}=d(i_j\Omega)=\mathrm{div}_\Omega j=\left(\sqrt{|g|}\,j^\mu\right)_{,\mu}d^4x$$

Here:

$$\left(\sqrt{|g|}\,j^\mu\right)_{,\mu}=\Big(\sqrt{|g|}\,\overbrace{g^{\mu\nu}(f\partial_\nu h-h\partial_\nu f)}^{\text{(is definition of } j)}\Big)_{,\mu}$$

$$=\cancel{\sqrt{|g|}\,g^{\mu\nu}\partial_\nu h\,\partial_\mu f}+f\overbrace{\left(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu h\right)_{,\mu}}^{=-m^2h\sqrt{|g|}}$$

$$-\cancel{\sqrt{|g|}\,g^{\mu\nu}\partial_\mu h\,\partial_\nu f}-h\overbrace{\left(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu f\right)_{,\mu}}^{=-m^2f\sqrt{|g|}}$$

$$=\sqrt{|g|}\left(-f\,m^2h+h\,m^2f\right)=0 \quad ✓$$

Recall:

$(\Box+m^2)\phi=0$ reads:

$\frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\phi\right)_{,\mu}+m^2\phi=0$

and the f and h are solutions of the K.G. eqn!

$\rightsquigarrow$ We finally proved that, for globally hyperbolic spacetimes, there always exist mode functions $\{u_k(x,t)\}$ so that our ansatz for $\hat{\phi}$ and $\hat{\pi}$ also obeys the CCRs at all time and indeed $\forall \Sigma$:

$$\sqrt{|g|}\, g^{0\nu} \int \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) d^3k = i\delta^3(x-x') \qquad (W)$$

Example: For Minkowski space, we had found this solution for the noninteracting Klein Gordon field:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_k\, e^{-i\omega_k t + ikx} + a_k^+ e^{i\omega_k t - ikx} \right) d^3k$$

We read off: $u_k(x,t) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t + ikx}$

Now: Verify the CCR condition, (W):

- Here: $\sqrt{|g|} = 1$ and $g^{0\nu} = \delta_{0\nu}$.
- Thus, the LHS of Eq. (W) reads:

$$\int u_k(x,t) \frac{\partial}{\partial x'^0} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^0} u_k(x',t)\ d^3k$$

$$= \frac{1}{(2\pi)^3} \int \frac{1}{2\omega_k} \Big[e^{-i\omega_k t + ikx} (i\omega_k) e^{i\omega_k t - ikx'}$$

$$- e^{i\omega_k t - ikx} (-i\omega_k) e^{-i\omega_k t + ikx'} \Big] d^3k$$

$$= \frac{1}{(2\pi)^3} \int \frac{2i\omega_k}{2\omega_k} e^{ik(x-x')} d^3k \overset{\text{Fourier}}{=} i\delta^3(x-x') \checkmark$$

Summary so far:

- To solve the QFT of a free KG field on curved spacetime is to solve the HC, EoM and CCRs.

- Make solution ansatz:

or integral, e.g. if no IR cutoff

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger \qquad (A)$$

- We showed that at least if spacetime is globally hyperbolic:

- There exists a set of solutions of the KG eqn, $\{u_k\}$, so that ansatz (A) solves HC, EoM and CCR for all time.

Q: Does there exist only one such set of solutions?

A: No, there exist many other such sets of solutions: $\{\bar{u}_k\}, \{\bar{\bar{u}}_k\} \ldots$

How to see this non-uniqueness?

- Recall symplectic form for $f, h \in V$:

$$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu}\left(f \partial_\nu h - h \partial_\nu f\right)$$

- Darboux: There exists a basis $\{v_n\}$ of V in which the form $(\ ,\ )$ reads:

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & 0 \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ 0 & & & \ddots \end{pmatrix}$$

- From the v_n we constructed the $u_n := v_{2n} + i\, v_{2n+1}$

- However: Darboux bases are not unique!

- Example: 2-dim. solution subspace.

Assume: $v_1, v_2 \in V$ are a Darboux basis, so

so that $(\,,\,)$ reads $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Q: Can we change basis, $v_i = Q\bar{v}_i$, so that $(\,,\,)$ keeps that matrix form? Is there a matrix Q so that

$$\left(\begin{array}{l} \text{Since } (v,w) = v^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w \text{ we require} \\ v^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w = \bar{v}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{w} \quad \forall \bar{v}, \bar{w} \\ (B\bar{v})^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (B\bar{w}) = \bar{v}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{w} \quad \forall \bar{v}, \bar{w} \\ \text{i.e.: } B^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \right) \rightarrow \qquad Q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ?$$

A: Yes, any change of basis $Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

with $ad - bc = 1$ will do. (Exercise: check)

- More generally: Also different boxes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ can get mixed!

Exercise: - Recall that with respect to the hermitean bi-linear form $\langle\,,\,\rangle$ of Lecture 12, the mode functions $\{u_n\}$ obey:

$$\left.\begin{array}{l} \langle u_n, u_m \rangle = \delta_{nm} \\ \langle u_n^*, u_m^* \rangle = -\delta_{nm} \\ \langle u_n, u_m^* \rangle = 0 = \langle u_n^*, u_m \rangle \end{array}\right\} (*)$$

- Now consider an invertible change of basis (in the space of complex-number-valued solutions of the K.G. eqn) to new mode functions:

$$\bar{u}_n := \sum_m \left(A_{nm} u_m + B_{nm} u_m^* \right)$$

- Show that for the $\{\bar{u}_n\}$ to qualify as mode functions, i.e., for them to obey $(*)$, i.e., $\langle \bar{u}_n, \bar{u}_m \rangle = \delta_{nm}$ etc, A, B must obey:

$$A^+ A - B^t B^* = 1 \text{ and } A^+ B - B^t A^* = 0$$

Non-uniqueness of the solution (A):

- Clearly, this means that there are infinitely many solutions of the form of (A) to HC, EoM and CCRs:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

$$\hat{\phi}(x,t) := \sum_k \bar{u}_k(x,t)\, \bar{a}_k + \bar{u}_k^*(x,t)\, \bar{a}_k^+$$

$$\hat{\phi}(x,t) := \sum_k \bar{\bar{u}}_k(x,t)\, \bar{\bar{a}}_k + \bar{\bar{u}}_k^*(x,t)\, \bar{\bar{a}}_k^+ \quad \text{etc, etc...}$$

- Correspondingly, we obtain different Fock bases:

Either: $a_k|0\rangle = 0 \qquad |n_k\rangle := \frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle$

Or: $\bar{a}_k|\bar{0}\rangle = 0 \qquad |\bar{n}_k\rangle := \frac{1}{\sqrt{n!}}(\bar{a}_k^+)^n|\bar{0}\rangle$ etc, etc...

- Q: Do these solutions of the QFT
  - describe different physics, or
  - do they differ by a mere change of basis in Fock space and so describe the same physics?

- A: It depends!

1) Assume first we can impose IR and UV cutoffs with negligible consequences

   * This means we truncate to a finite (though large) number of independent mode oscillators, a_n, a_n^+.

   * Then, the following theorem implies that all solutions to HC, EoM, CCR differ merely by a change of basis:

Theorem (Stone and von Neumann):

* Assume in a Hilbert space, $\mathcal{H}$, the operators $\hat{x}_i, \hat{p}_j$ obey:

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \qquad [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j] \qquad i,j \in \{1,\dots,N\}$$

* Assume that in a Hilbert space $\tilde{\mathcal{H}}$ other operators $\tilde{x}_i, \tilde{p}_j$ also obey:

$$[\tilde{x}_i, \tilde{p}_j] = i\delta_{ij} \qquad [\tilde{x}_i, \tilde{x}_j] = 0 = [\tilde{p}_i, \tilde{p}_j] \qquad i,j \in \{1,\dots,N\}$$

* Assume that the representations are irreducible (i.e., no invariant subspace)

We can assume that $\mathcal{H}' = \mathcal{H}$ because all separable Hilbert spaces (the usual: with countable bases) are unitarily equivalent.

Then: there exists a unitary operator $\hat{U}$ so that:

$$\tilde{x}_i = U \hat{x}_i U^+ \qquad \tilde{p}_i = U \hat{p}_i U^+ \quad \text{(i.e., a change of basis)}$$

* Remark: Strictly speaking, there can be pathological cases. The pathological cases can be avoided by requiring representations of the CCRs of the (bounded and therefore better behaved) operators:

$$e^{i\alpha_i \hat{x}_i}, \quad e^{i\beta_i \hat{p}_i}$$

* Application to QM and to UV & IR regularized QFT:

Consider $\hat{x}_n := \frac{1}{\sqrt{2}}(a_n + a_n^+)$, $\hat{p}_n := \frac{-i}{\sqrt{2}}(a_n - a_n^+)$

and $\bar{x}_n := \frac{1}{\sqrt{2}}(\bar{a}_n + \bar{a}_n^+)$, $\bar{p}_n := \frac{-i}{\sqrt{2}}(\bar{a}_n - \bar{a}_n^+)$ etc.

The theorem of Stone & v. Neumann implies that

$$a_n = \hat{U} \bar{a}_n \hat{U} \quad \text{with } \hat{U} \text{ unitary.}$$

$\Rightarrow$ All solutions are the same up to a mere change of basis.

2.) Consider now the possibility that we cannot truncate to a finite number of degrees of freedom.

Q: When would this happen?

A: E.g., phase transitions formally need systems with an infinite number of degrees of freedom.

Then: The QFT can have unitarily non-equivalent solutions, that differ physically: different "phases".

Underlying math of non-equivalent representations?

Assume $\langle a|b\rangle = \alpha$ with $0<\alpha<1$, i.e., not $\perp$

Then $(\langle a|\langle a|\langle a|\ldots\langle a|)\overbrace{(|b\rangle|b\rangle|b\rangle\ldots|b\rangle)}^{N} = \alpha^N$, i.e., not $\perp$

But for $N=\infty$ have $|a\rangle|a\rangle\ldots|a\rangle \perp |b\rangle|b\rangle\ldots|b\rangle$, so that then can no longer use $|a\rangle|a\rangle\ldots|a\rangle$ to help linearly combine, e.g., $|b\rangle|b\rangle\ldots|b\rangle$.

From now on: We will assume IR & UV cutoffs are possible and that Stone v. Neumann therefore applies.

Therefore:

- No matter which set of suitable mode functions

$$\{u_n(x,t)\} \text{ or } \{\bar{u}_n(x,t)\} \text{ or } \{\bar{\bar{u}}_n(x,t)\}, \ldots$$

we choose, we obtain the same solution

$$\hat{\phi}(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$
$$= \sum_k \bar{u}_k(x,t)\, \bar{a}_k + \bar{u}_k^*(x,t)\, \bar{a}_k^+$$
$$= \sum_k \bar{\bar{u}}_k(x,t)\, \bar{\bar{a}}_k + \bar{\bar{u}}_k^*(x,t)\, \bar{\bar{a}}_k^+ \quad = \ldots$$

with their Fock bases being different bases in the same Hilbert space.

- For example, using the $\{u_k\}$, we are led to span the Hilbert space $\mathcal{H}$ using this ON basis:

$$|0\rangle \text{ where } a_k|0\rangle = 0 \quad \forall k$$
$$a_k^+|0\rangle, \ \frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle$$
$$\frac{1}{\sqrt{\cdots}}(a_k^+)^{\cdot\cdot}\cdots(a_{k'}^+)^{\cdot\cdot}|0\rangle, \text{ etc} \ldots$$

- Or, using other mode functions, say $\{\bar{u}_k\}$, we may span the same Hilbert space, $\mathcal{H}$, using this ON basis:

$$|\bar{0}\rangle \text{ where } \bar{a}_k|\bar{0}\rangle = 0 \quad \forall k$$
$$\bar{a}_k^+|\bar{0}\rangle, \ \frac{1}{\sqrt{n!}}(\bar{a}_k^+)^n|\bar{0}\rangle$$
$$\frac{1}{\sqrt{\cdots}}(\bar{a}_k^+)^{\cdot\cdot}\cdots(\bar{a}_{k'}^+)^{\cdot\cdot}|\bar{0}\rangle, \text{ etc} \ldots$$

Does the choice of mode functions matter?

- In principle, it does not:

Any state of the system, say $|\Psi\rangle$, can be expanded in each basis.

- In practice, however:

It is convenient, whenever we know which state is the no-particle (i.e., vacuum) state, say $|\Omega\rangle$, to choose the mode functions $\{u_k\}$ such that the corresponding $|0\rangle$ is $|\Omega\rangle$, i.e., such that

$$|0\rangle = |\Omega\rangle, \text{ i.e., such that } a_k|\Omega\rangle = 0$$

Then, conveniently, states like $\frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle$ are the multi-particle states.

Outlook: (only a rough sketch)

- Say we know the system's state, $|\Psi\rangle$, is the vacuum initially.
- $\rightsquigarrow$ We choose $\{u_n\}$ appropriately, so that $|0\rangle_{in} = |\Psi\rangle$.
- After some evolution (e.g. the universe expands) the vacuum state may be a different state, say $|\chi\rangle$.
- $\rightsquigarrow$ We choose $\{\bar{u}_n\}$ appropriately, so that $|\bar{0}\rangle_{out} = |\chi\rangle$
- At late times, since we work in the Heisenberg picture, the system is still in the state $|0_{in}\rangle$, but this is then an excited state!

  $\rightsquigarrow$ Description of particle production due to cosmic expansion.
- **Recall:** We had an analogous situation with driven harmonic oscillators!