QFT for Cosmology, Achim Kempf, Lecture 14

Note Title

Quantum field theory on FRW spacetimes.

Observations:

On scales > 1 GLy:

- The universe is spatially very flat
- The cosmic expansion is very isotropic.

Friedmann Robertson Walker (FRW) spacetimes:

- Simplifying approximation:

Spacetime is modeled as having

- no spatial curvature at all.
- entirely isotropic expansion

(Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

There are even solutions that only temporarily get very close to flatness. The Einstein eqns are nonlinear!)

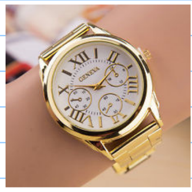
With these assumptions, we choose convenient coordinates:

* Time coordinate t:

Definition: The motion of galaxies due to the cosmic expansion is called the Hubble flow.

Definition: The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

Definition: As the time coordinate, t, let us use the proper time, t, of a freely streaming observer who has no peculiar velocity.



(to a good approximation, you can use your wrist watch on earth)

* Space coordinates:

It is convenient to use "comoving coordinates", x_1, x_2, x_3:

- At one time, t_0, (say today) we set up an ordinary rectangular coordinate system.
- Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

Advantages:

- In the comoving coordinate system, galaxies have constant coordinates, except for possible peculiar motion.
- Waves keep their wave lengths numerically constant even while they get physically stretched.

* The metric:

Recall that $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

is the invariant 4-distance.

In our coordinates, $g_{\mu\nu}(x)$ must read:

because we use wrist watch "proper" time

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$$

because our coordinate system's unit of length means over time a larger and larger proper length.

* The "scale factor":

- The scale factor function $a(t)$ is needed to take into account the expansion when calculating distances.
  - Example: The proper distance d between two galaxies with comoving distance $(\Delta x_1, \Delta x_2, \Delta x_3)$ at proper time t is:

$$d = \sqrt{|g_{\mu\nu}(t_0)\, \Delta x^\mu \Delta x^\nu|}$$

$$= a(t) \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

Note: $\Delta x_0 = t_0 - t_0 = 0$ since we are looking at the distance between the galaxies at equal time.

* Dynamics of $a(t)$:

The function $a(t)$ is determined by **all** equations of motion:

1. Calculate the energy momentum tensor $T_{\mu\nu}(t,\vec{x})$ contributions of at least the most important fields, say $\varphi_i(x,t)$.

2. Solve, simultaneously:

   * The equations of motion for the fields φ_i

   * The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(t,x) = \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix}$:

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G \, T_{\mu\nu}(x)$$

* Semi-classical approximation

We can solve these classically, but not quantum mechanically:

Can quantize only φ_i, not $g_{\mu\nu}$.

$\Rightarrow$ need to "make quantum $T_{\mu\nu}(t,\vec{x})$ classical" for Einstein eqn!

$\rightsquigarrow$ One uses: $\quad \bar{T}_{\mu\nu}(x) := \langle \Omega | T_{\mu\nu}(t,\vec{x}) | \Omega \rangle$

Problem: Energy & Momentum are naturally nonlocal because of uncertainty principle.

Remark: $\dot{a}(t)$ is related to curvature between space & time.

For now, we will assume that the expansion's scale factor function $a(t)$ is given.

Convenient Definition: The conformal time coordinate, η.

- Recall that:

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & 0 \\ & -a^2(t) & & \\ & & -a^2(t) & \\ 0 & & & -a^2(t) \end{pmatrix}$$

- It would be convenient if $g_{\mu\nu}$ were proportional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

- This can be achieved by choosing a new time coordinate η, so that time also has a prefactor a^2, i.e., so that:

$$(\Delta t)^2 = a^2(t_0)(\Delta\eta)^2$$

- To this end, we need: $a\, d\eta = dt$

i.e.: $\frac{d\eta}{dt} = \frac{1}{a}$

and therefore $\eta(t) = \int_{t_0}^{t} \frac{1}{a(t')}\, dt'$

t_0 ← yields arbitrary integration constant.

- The variable η is called the "conformal time".

(...because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale-factor-only transformations)

- Using conformal time and comoving spatial coordinates the metric reads:

$$g_{\mu\nu}(\eta,\vec{x}) = a^2(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \underbrace{a^2(\eta)\ \eta_{\mu\nu}}_{\text{do not mix up}}$$

- This also implies:

$$g^{\mu\nu}(\eta,\vec{x}) = a^{-2}(\eta)\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix} = a^{-2}(\eta)\,\eta^{\mu\nu}$$

Recall: $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$, i.e., $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverse to another.

- We easily obtain the integral measure needed for the action:

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu}(\eta,\vec{x}))|} = a^4(\eta)$$

The Klein Gordon field in FRW spacetimes

- Neglecting a potential $V(\phi)$ for now, we obtain the action of the "free K.G. field on the FRW background":

$$S_{KG} = \int\left(\frac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)\sqrt{|g|}\,d^4x$$

$$\overset{\text{here}}{=} \int\left(\frac{1}{2}a^{-2}(\eta)\,\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)a^4\,d\eta\,d^3x$$

- Thus, from the general Euler Lagrange equation

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

$$\left(\frac{1}{a^4(\eta)}\frac{\partial}{\partial x^\mu}\eta^{\mu\nu}a^2\frac{\partial}{\partial x^\nu}+m^2\right)\phi(x)=0$$

$a' = \frac{da}{d\eta}$

$$\left(\frac{1}{a^4(\eta)}\eta^{\mu\nu}a^2(\eta)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}+\frac{1}{a^4(\eta)}2a'a\frac{\partial}{\partial x^0}+m^2\right)\phi(x)=0$$

$$\boxed{\phi''(\eta,\vec{x})+2\frac{a'(\eta)}{a(\eta)}\phi'(\eta,\vec{x})-\Delta\phi(\eta,\vec{x})+a^2(\eta)m^2\phi(\eta,\vec{x})=0}$$

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi''+\text{※}\phi'+\text{※}\phi=0$$

a time-dependent friction-like term that is entirely new.

a term that also occurs in the usual harmonic oscillator. Notice though that it is now time-dependent.

Strategy: Use a new, re-scaled, field variable χ:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

This simple ansatz succeeds:

$$\chi(\eta,\vec{x}):=a(\eta)\,\phi(\eta,\vec{x})$$

Namely:

we have: $\phi'=\frac{\partial}{\partial\eta}\frac{1}{a}\chi=-\frac{a'}{a^2}\chi+\frac{1}{a}\chi'$

and: $\phi_{,i}=\frac{\partial}{\partial x^i}\frac{1}{a(\eta)}\chi(\eta,\vec{x})=\frac{1}{a}\chi_{,i}$ for $i=1,2,3$

Using these, the action in terms of χ becomes:

$$S_{\chi G} = \int \frac{1}{2}\Big(\chi'^2 - \sum_{i=1}^{3} \chi_{,i}^2 - \underbrace{\Big(m^2a^2 - \frac{a''}{a}\Big)}\chi^2\Big)\, d\eta\, d^3x$$

Note that this term is like a time-dependent mass term $m_{eff}^2(\eta)$

Exercise: verify

Equation of motion:

* Do

$$\frac{\delta S}{\delta \phi(\eta,\vec{x})} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \chi(\eta,\vec{x})} = 0$$

yield equivalent equations of motion?

* Yes, because:

$$0 = \frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta \chi}\,\frac{\delta \chi}{\delta \phi}$$

if $\delta S/\delta\chi$ vanishes then also $\delta S/\delta\phi$ vanishes.

* Thus, we may calculate the equation of motion directly in terms of χ from $S[\chi]$, to obtain:

Exercise: verify!

$$\chi'' - \Delta\chi + \Big(m^2a^2 - \frac{a''}{a}\Big)\chi = 0 \qquad \text{(EoM 1)}$$

Remark:

We could have obtained this equation of motion directly from that of ϕ by change of variable. But finding the action for χ was still worthwhile, namely to get the conjugate to χ!

Preparation for quantization:

* We need the canonically conjugate field

$$\pi^{(x)}(\eta,\vec{x})$$

to the field $x(\eta,\vec{x})$, i.e., the **Legendre transform of x'**.

* To this end, we consider the Lagrangian:

$$L = \int \frac{1}{2}\left(x'^2 - \sum_{i=1}^{3} x_{,i}{}^2 - \left(m^2a^2 - \frac{a''}{a}\right)x\right) d^3x$$

* Thus, the Legendre transformed variable reads:

$$\pi^{(x)}(\eta,\vec{x}) := \frac{\delta L}{\delta x'(\eta,\vec{x})} = x'(\eta,\vec{x}) \qquad \text{(EoM 2)}$$

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* Which is the field that is conjugate to ϕ?

$$S_{K.G.} = \int\left(\frac{1}{2}a^{-2}(\eta)\,\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)a^4\,d\eta\,d^3x$$

$\Rightarrow$ The field $\pi^{(\phi)}$ which is conjugate to ϕ reads:

$$\pi^{(\phi)} := \frac{\delta L}{\delta \phi'} = a^2\phi'$$

* Compare:

$$\begin{aligned}\pi^{(x)} &= x' \\ &= (a\phi)' \\ &= a\phi' + a'\phi \\ &= \frac{1}{a}\pi^{(\phi)} + a'\phi\end{aligned}$$

, i.e., $\pi^{(\phi)}$, $\pi^{(x)}$ are different!

Quantization:

$$[\hat{\phi}(\eta,\vec{x}),\hat{\pi}^{(\phi)}(\eta,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{\phi}(\eta,\vec{x}),\hat{\phi}(\eta,\vec{x}')] = 0$$

$$[\hat{\pi}^{(\phi)}(\eta,\vec{x}),\hat{\pi}^{(\phi)}(\eta,\vec{x}')] = 0$$

Proposition:

In terms of the fields $\hat{\chi} := a\,\hat{\phi}$, $\hat{\pi}^{(\chi)} := \hat{\chi}'$, these commutation relations become:

$$[\hat{\chi}(\eta,\vec{x}),\hat{\pi}^{(\chi)}(\eta,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{\chi}(\eta,\vec{x}),\hat{\chi}(\eta,\vec{x}')] = 0$$

$$[\hat{\pi}^{(\chi)}(\eta,\vec{x}),\hat{\pi}^{(\chi)}(\eta,\vec{x}')] = 0$$

Proof: Only the first CCR is nontrivial to check:

$$[\hat{\chi}(\eta,\vec{x}),\hat{\pi}^{(\chi)}(\eta,\vec{x}')] = \left[a(\eta)\,\hat{\phi}(\eta,\vec{x}),\ \frac{1}{a(\eta)}\hat{\pi}^{(\phi)}(\eta,\vec{x}')+a'(\eta)\hat{\phi}(\eta,\vec{x}')\right]$$

$$= \left[\hat{\phi}(\eta,\vec{x}),\hat{\pi}^{(\phi)}(\eta,\vec{x}')\right]$$

$$= i\delta^3(\vec{x}-\vec{x}')$$

Thus, the change from ϕ to χ is fairly trivial.

Notice, however:

$$L \xrightarrow[\phi' \text{ replaced by } \pi^{\phi}]{\text{L.T.}} H^{(\phi)} := \int \phi'\,\pi^{(\phi)}\,d^3x - L$$

$$L \xrightarrow[\chi' \text{ replaced by } \pi^{\chi}]{\text{L.T.}} H^{(\pi)} := \int \chi'\,\pi^{(\chi)}\,d^3x - L$$

they have no reason to be the same!

Question:

How can both be valid generators of time evolution,

i.e., how can we have:

$$i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}] \quad \text{and} \quad i\,\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

and yet $\hat{H}^{(\phi)} \neq \hat{H}^{(x)}$?

- Should there not be one Hamiltonian for all variables?

Answer: Yes, and it is, of course $\hat{H}^{(\phi)}$.

This extra term is there if the variable $\hat{Q}$ has also explicit time-dependence, e.g., $\hat{Q} = \cos(at+bt^2)\hat{q} + c\hat{p}$, or here: $\hat{x} = \frac{1}{a}\hat{\phi}$.

Recall that in QM: $i\hat{Q}' = [\hat{Q}, \hat{H}] + i\frac{\partial}{\partial t}\hat{Q}$

Explicitly:

* From $\hat{x} = a\hat{\phi}$ and $i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}]$ we obtain:

$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\frac{1}{a}\hat{x}' - i\frac{a'}{a^2}\hat{x} = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\,\hat{x}' = [\hat{x}, \hat{H}^{(\phi)}] + i\frac{a'}{a}\hat{x}$$

* But we also have:

$$i\,\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

$\Rightarrow$ We must have: $\quad \hat{H}^{(x)} \neq \hat{H}^{(\phi)}$

Since there are multiple Hamiltonians, which, if anyone, is the energy?

- One usually defines the energy as the generator of time evolution. We saw that in the presence of gravity this is ambiguous: one can define many different Hamiltonians for the same theory (same action).

- Therefore, with Einstein, we define the energy (density) not as the generator of time evolution but as a generator of curvature:

- **Recall:** The Einstein equation

$$\underbrace{R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x)}_{\text{curvature}} = \underbrace{8\pi G\, T_{\mu\nu}(x)}_{\text{"energy momentum"}}$$

- **Recall:** The K.G. field's energy-momentum tensor

$$T^{KG}_{\mu\nu}(\eta,\vec{x}) = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2} m^2 \phi^2\right]$$

- Consider $T_{00}(\eta,\vec{x})$, which is called the "**energy density**":

Note: In differential geometry, there is also another use of the term "density": For every tensor, say $Q_{\mu\nu}$, there is a so-called "tensor density" $\tilde{Q}_{\mu\nu}$, defined as $\tilde{Q}_{\mu\nu} := Q_{\mu\nu}\sqrt{g}$, which absorbs the obligatory measure factor in integrations.

$$T_{00}(\eta,\vec{x}) = a^{-4}\frac{1}{2}\pi^{(\phi)\,2} + \frac{1}{2}\sum_{i=1}^{3}\phi_{,i}^{\,2} + \frac{a^2}{2} m^2 \phi^2 \qquad \text{(T)}$$

- **Exercises:**

  a) Verify (T).

  b) Calculate $H^{(\phi)}$.

  Notice that $H^{(\phi)}$ is not a scalar.

  c) Show that $H^{(\phi)}(\eta) = \int_{\mathbb{R}^3} T^0_{\ 0}(\eta,\vec{x})\sqrt{g}\, d^3x$.

  d) Calculate $H^{(x)}(\eta)$.