

Solving the quantized K.G. eqn. on FRW spacetimes

Recall:

1.) We obtain the solution  $\hat{\phi}(x,t)$  through the ansatz

$$\hat{\phi}(x,t) = \sum_k u_k(x,t) a_k + u_k^*(x,t) a_k^\dagger \quad (*)$$

\* if we use operators  $a_k$  obeying  $[a_k, a_{k'}^\dagger] = \delta^3(k-k')$  and

\* if we find classical solutions  $\{u_k(x,t)\}$  of the K.G. eqn., called mode functions, which obey:

$$\sqrt{|g|} g^{\alpha\beta} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x^\alpha} u_k^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^\alpha} u_k(x,t) \right) = i \delta^3(\vec{x} - \vec{x}') \quad (C)$$

2.) Then, we can use the  $\{a_k\}$  to build a convenient basis in the Hilbert space:

□ Namely:  $|0\rangle$  is the vector obeying  $a_k |0\rangle = 0$

□ The other basis vectors are:

$$a_k^\dagger |0\rangle, \dots, \frac{1}{\sqrt{n!}} (a_k^\dagger)^n |0\rangle, \dots, a_k^\dagger a_{k'}^\dagger |0\rangle, \dots$$

$$\frac{1}{\sqrt{n_1!}} \dots \frac{1}{\sqrt{n_m!}} (a_{k_1}^\dagger)^{n_1} \dots (a_{k_m}^\dagger)^{n_m} |0\rangle, \dots \text{ etc.}$$

3.) Choosing a different set of classical solutions  $\{\tilde{u}_k(x,t)\}$  which obey (C) yields the same  $\hat{\phi}(x,t)$ , namely

$$\hat{\phi}(x,t) = \sum_k \tilde{u}_k(x,t) \tilde{a}_k + \tilde{u}_k^*(x,t) \tilde{a}_k^\dagger$$

but the basis of vectors  $|\tilde{0}\rangle, \tilde{a}_k^\dagger |\tilde{0}\rangle, \tilde{a}_k^\dagger \tilde{a}_{k'}^\dagger |\tilde{0}\rangle \dots$  is a different basis. Recall: Stone von Neumann theorem.

## Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for  $\hat{\phi}(x,t)$  directly, but instead for:

$$\hat{\chi}(\eta, x) := a(\eta) \hat{\phi}(\eta, x)$$

- In terms of  $\hat{\chi}(\eta, x)$  the quantum K.G. eqn. reads:

$$\hat{\chi}''(\eta, x) - \Delta \hat{\chi}(\eta, x) + \left( m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right) \hat{\chi}(\eta, x) = 0$$

- Note: This is a partial differential equation because both time and space derivatives occur.

- Observation: The derivatives  $\frac{\partial}{\partial x^i}$  become multiplication operators  $ik_i$  under spatial Fourier transform.

- Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

- Define:  $\hat{\chi}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}(\eta, x) e^{-ikx} d^3x$

$$\text{i.e.: } \hat{\chi}(\eta, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}_k(\eta) e^{ikx} d^3k$$

- Analogously:

$$\hat{\pi}_k^{(\text{cc})}(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\pi}^{(\text{cc})}(\eta, x) e^{-ikx} d^3x$$

□ Thus, in terms of  $\hat{\chi}_k(\eta)$ , the K.G. eqn. reads:

$$\hat{\chi}_k''(\eta) + \left( k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right) \hat{\chi}_k(\eta) = 0 \quad (\text{EoM})$$

⇒ for each **comoving** Fourier mode  $k$  the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta) \hat{\chi}_k(\eta) = 0$$

$$\text{with: } \omega_k(\eta) := \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$$

Remark: It is not surprising that the frequency of each comoving mode is changing because its physical wavelength is changing too.

### In extreme cases:

The frequency  $\omega_k(\eta)$  may become imaginary, namely if  $a''(\eta)$  is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on  $k$ , i.e., some modes may have imaginary frequencies while others don't.

### □ Exercise:

\* Show that  $\hat{\chi}_k^+(\eta) = \hat{\chi}_{-k}(\eta)$ ,  $\hat{\pi}_k^{(\mu)+}(\eta) = \hat{\pi}_{-k}^{(\mu)}(\eta)$  (HC)

\* Show that

$$[\hat{\chi}_k(\eta), \hat{\pi}_{k'}(\eta)] = i \delta^3(k+k') \quad (\text{CCR})$$

$$\text{i.e. } [\hat{\chi}_k(\eta), \hat{\pi}_{k'}^+(\eta)] = i \delta^3(k-k')$$

- In order to solve EoM, HC, CCR for  $x_k(\eta)$ , we make this ansatz:

$$\hat{x}_k(\eta) := \frac{1}{\sqrt{2}} \left( v_k^*(\eta) a_k + v_k(\eta) a_k^\dagger \right) \quad (A)$$

← convenient later

- Exercise: Express the mode functions  $u_k(\eta, x)$  of (\*) in terms of the functions  $v_k(\eta)$ .

□ Proposition: The ansatz (A)

- 1.) solves the hermiticity condition (HC) by construction.
- 2.) solves the (EoM), if the  $v_k(\eta)$  each solve (EoM) as (complex!) number-valued functions:

(Note: The equation depends only on  $|k|$ , not on the direction of  $k$ . Thus if  $v_k(\eta)$  is a solution for one  $k$  then it is solution for all  $k'$  with  $|k'| = |k|$ .  $\Rightarrow$  We can and will choose  $v_k(\eta) = v_{|k|}(\eta)$ )

$$v_k''(\eta) + \left( k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right) v_k(\eta) = 0 \quad (M)$$

- 3.) the commutation relations (CCR) if the  $v_k$  are chosen such that they also obey:

$$v_k'(\eta) v_k^*(\eta) - v_k(\eta) v_k'^*(\eta) = 2i \quad (W)$$

□ Exercise:

- a) Prove the proposition.
- b) Assume that  $v_k(\eta)$  is any solution of (EoM). Show that if (W) holds at one time then it holds at all time.

(Note: The LHS of (W) is the "Wronskian" of the ODE)

## Conclusion:

In order to obtain the solution  $\hat{\phi}(\gamma, x)$ , we do:

- A) Find for each mode  $k \in \mathbb{R}^3$  a solution  $v_k(\gamma)$  to (M), i.e., a solution to the classical harmonic oscillator with time-dependent frequency.
- B) Make sure  $v_k(\gamma)$  obeys (W), if need be by multiplying with a constant. (Recall exercise b))
- C) Build a basis in the Hilbert space:  
 $a_k |0\rangle = 0$ ,  $a_k^\dagger |0\rangle$ ,  $a_k^\dagger a_k^\dagger |0\rangle$ , etc...

## Choice of mode solutions $\{v_k(\gamma)\}$

- For each choice, say  $\{v_k(\gamma)\}_{k \in \mathbb{R}^3}$  or  $\{\tilde{v}_k(\gamma)\}_{k \in \mathbb{R}^3}$  we obtain the same  $\hat{\phi}(x, t)$  but the bases

$$|0\rangle, a_k^\dagger |0\rangle, a_k^\dagger a_k^\dagger |0\rangle, \dots$$

and

$$|\tilde{0}\rangle, \tilde{a}_k^\dagger |\tilde{0}\rangle, \tilde{a}_k^\dagger \tilde{a}_k^\dagger |\tilde{0}\rangle, \dots$$

will of course be different.

- We will often find it convenient to use the basis  $|0\rangle, a_k^\dagger |0\rangle, a_k^\dagger a_k^\dagger |0\rangle, \dots$  that comes with one set of mode functions  $\{v_k(\gamma)\}$  at one time (say initially) and then the basis  $|\tilde{0}\rangle, \tilde{a}_k^\dagger |\tilde{0}\rangle, \tilde{a}_k^\dagger \tilde{a}_k^\dagger |\tilde{0}\rangle, \dots$  of some  $\{\tilde{v}_k(\gamma)\}$  later.

Why?

□ In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.

□ But the observables evolve in time!

⇒ The meanings of all Hilbert space vectors change over time

⇒ We may, e.g., choose a set  $\{v_k\}$  whose vector  $|0\rangle$  happens to be the vacuum state at one time and we may choose another set  $\{\tilde{v}_k\}$  whose vector  $|\tilde{0}\rangle$  happens to be the vacuum state at another time.

□ How many possible choices of

$$\{v_k(\eta)\}_{k \in \mathbb{R}^3}, \{\tilde{v}_k(\eta)\}_{k \in \mathbb{R}^3}, \{\hat{\tilde{v}}_k(\eta)\}_{k \in \mathbb{R}^3}, \dots$$

do exist?

□ Let us consider each mode,  $k$ , separately:

□ The solution space of (M), for fixed  $k$ ,

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

Note: Every solution obeying (M) must be complex-valued. Why?

□ If  $v_k$  is a complex-valued solution, then

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.



Every solution,  $\tilde{v}_k$ , is a linear combination of  $v_k, v_k^*$ , i.e., there must exist  $\alpha, \beta \in \mathbb{C}$ , so that:

$$\tilde{v}_k(\gamma) = \alpha_k v_k(\gamma) + \beta_k v_k^*(\gamma)$$

□ The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions  $v_k, \tilde{v}_k$ , etc must also obey (W), i.e.:

$$v_k'(\gamma) v_k^*(\gamma) - v_k(\gamma) v_k'^*(\gamma) = 2i \quad (W)$$

(i.e.  $\text{Im}(v'v^*) = 1$ , which is only one real equation)

□ Proposition:

Assume  $v_k$  obeys (W). Then,  $\tilde{v}_k$  defined through

$$\tilde{v}_k(\gamma) = \alpha_k v_k(\gamma) + \beta_k v_k^*(\gamma) \quad (B)$$

also obeys (W), iff the coefficients  $\alpha_k, \beta_k$  obey:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

□ Proof: Exercise

$$\begin{aligned} \Rightarrow \text{we easily obtain: } \mathcal{C}_k(\gamma) &= \frac{1}{\sqrt{2}} (v_k^*(\gamma) a_k + v_k(\gamma) a_{-k}^*) \\ &= \frac{1}{\sqrt{2}} (\tilde{v}_k^*(\gamma) \tilde{a}_k + \tilde{v}_k(\gamma) \tilde{a}_{-k}^*) \\ &= \dots \end{aligned} \quad (P)$$

□ Terminology: Such a transformation from one choice  $\{v_k\}$ ,  $a_k$  and corresponding basis

$$|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^+ |0\rangle, \dots$$

to some  $\{\tilde{v}_k\}$ ,  $|\tilde{0}\rangle$ ,  $\tilde{a}_k$  and their basis

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$$

is called a "Bogolubov transformation".

Strategy: We have two tasks now:

\* Make Bogolubov Hilbert basis transforms explicit.

(E.g. so that  $|0\rangle$  is, at least at one time, the vacuum.)

\* Find out when which choice of  $\{v_k\}$  is convenient.

## Bogolubov transformations of Hilbert bases

□ How can we express the basis vectors

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \frac{1}{\sqrt{2!}} \tilde{a}_k^2 |\tilde{0}\rangle, \dots, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$$

as linear combinations of the basis

$$|0\rangle, a_k^+ |0\rangle, \frac{1}{\sqrt{2!}} a_k^2 |0\rangle, \dots, a_k^+ a_k^+ |0\rangle, \dots ?$$

□ Proposition: Equations (B) & (P) yield:  $a_k = d_k^+ \tilde{a}_k + \beta_k \tilde{a}_k^+$

Proof: Exercise.

□ Now we observe that  $a_k |0\rangle = 0$  becomes:

$$(d_k^+ \tilde{a}_k + \beta_k \tilde{a}_k^+) |0\rangle = 0$$



□ Try to solve for  $|0\rangle$  using ansatz:  $|0\rangle := \left( \prod_k f_k(\tilde{a}_k^+, \tilde{a}_{-k}^+) \right) |\tilde{0}\rangle$

□ Proposition:

$$|0\rangle = \left[ \prod_k \frac{1}{|d_k|^{1/2}} e^{-\frac{\beta_k}{2d_k^*} \tilde{a}_k^+ \tilde{a}_{-k}^+} \right] |\tilde{0}\rangle \quad (T)$$

← needed for normalization

□ Proof: Exercise.

Hint: Use  $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \dots$

### Interpretation of (T):

□ Assume, e.g., that  $|0\rangle$  and  $|\tilde{0}\rangle$  are those Hilbert space vectors which happen to be the vacuum state vectors at the times  $\eta_1$  and  $\eta_2$  respectively.

(We will soon explore how to identify the vacuum state at any given time)

□ Assume at time  $\eta_1$  the system's state  $|\Omega\rangle$  is the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions  $\{v_n\}$  so that:

$$|\Omega\rangle = |0\rangle$$

At a later time,  $\eta_2$ , the system is still in this state but the vacuum is then some other vector  $|\tilde{0}\rangle$ , which obeys  $\tilde{a}_n |\tilde{0}\rangle = 0$  with mode functions  $\tilde{v}_n(t)$ .

□ Thus, from (T) we see that the system's state,  $|\Omega\rangle$ , is at  $\eta_2$  a state with many particles.

□ Note: the particles have been created in  $k, -k$  pairs.

□ Intuitively: The expansion rips virtual particle + antiparticle pairs apart.