QFT for Cosmology, Achim Kempf, Lecture 16

Note Title

Recall:

- Using different choices of mode functions, $v_k(\eta)$, $\tilde{v}_k(\eta)$, we can write $\hat{x}_k(\eta)$ in different ways:

$$\hat{x}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right) \qquad (A)$$

$$= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right)$$

- Since for each k the space of possible mode functions is 2-dimensional (complex), there exist complex α_k, β_k so that:

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta) \qquad (B)$$

(Recall: Because $\tilde{v}_k(\eta)$ must obey the Wronskian condition, α_k and β_k must obey $|\alpha_k|^2 - |\beta_k|^2 = 1$)

- From (A) and (B) we obtain (exercise):

$$\boxed{a_k = \alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+}$$

- Thus, $a_k|0\rangle = 0$ becomes $(\alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+)|0\rangle = 0$, which yields:

$$|0\rangle = \left[\prod_k \underbrace{\frac{1}{|\alpha_k|^{1/2}}}_{\text{needed for normalization}} e^{-\frac{\beta_k}{2\alpha_k^*} \tilde{a}_k^+ \tilde{a}_{-k}^+}\right] |\tilde{0}\rangle \qquad (T)$$

$\Rightarrow$ We can now express all basis vectors $|0\rangle$, $a_k^+|0\rangle$, $a_k^+ a_{k'}^+|0\rangle \ldots$ in terms of the basis vectors $|\tilde{0}\rangle$, $\tilde{a}_k^+|\tilde{0}\rangle$, $\tilde{a}_k^+ \tilde{a}_{k'}^+|\tilde{0}\rangle \ldots$.

Example scenario:

- ✱ Assume $v_k(\eta)$, $\tilde{v}_k(\eta)$ chosen so that $|0\rangle$, $|\tilde{0}\rangle$ are vacuum at η_1, η_2.
- ✱ Assume system is in vacuum state at η_1, i.e. $|\Omega\rangle = |0\rangle$.
- ✱ Then system's state $|\Omega\rangle$ at η_2 is an excited state, i.e., a state with particles!

The extent of particle creation?

- Eqn. (T) shows that there is a finite probability amplitude for finding arbitrarily many particles at time t_2. Does that mean ∞ many get created (at ∞ energy expense and thus halting the expansion?)

- Let us calculate the expected number of created particles:

  * Definition (QM):
  $$\hat{N} := a^+ a \text{ is called a "Number operator"}$$
  * Why? It is a self-adjoint observable with eigenbasis:
  $$\hat{N}(a^+)^n|0\rangle = n\,(a^+)^n|0\rangle$$
  * Exercise: verify.

* Definition (QFT): $\hat{N}_k := a_k^+ a_k$

Interpretation of $\hat{N}_k$ in QFT

* Assume that at some time, η, the state $|0\rangle$ is the vacuum.

* Thus, at η, for example the state $(a_k^+)^n|0\rangle$ is a state with n particles of momentum k.

* Now assume that at η the system is in an arbitrary state $|\Omega\rangle$.

* Then, at η, the expected number of particles of momentum k is:

$$\overline{N}_k = \langle\Omega|\,\hat{N}_k\,|\Omega\rangle$$

Calculation in the above scenario for $\tilde{N}_k := \tilde{a}_k^+ \tilde{a}_k$ at time η_2

$$\bar{\tilde{N}}_k = \langle \Omega | \hat{\tilde{N}}_k | \Omega \rangle$$

$$= \langle 0 | \tilde{a}_k^+ \tilde{a}_k | 0 \rangle$$

Now use that $a_k = \alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+$, i.e. also, that $\tilde{a}_k = \tilde{\alpha}_k^* a_k + \tilde{\beta}_k a_{-k}^+$

Exercise: Calculate $\tilde{\alpha}_k, \tilde{\beta}_k$ in terms of α_k, β_k.

$$= \langle 0 | (\tilde{\alpha}_k a_k^+ + \tilde{\beta}_k^* a_{-k})(\tilde{\alpha}_k^* a_k + \tilde{\beta}_k a_{-k}^+) | 0 \rangle$$

$$= \langle 0 | \tilde{\beta}_k^* \tilde{\beta}_k a_{-k} a_{-k}^+ + * \cancel{a^+ a} + * \cancel{a a^+} + * \cancel{a a} | 0 \rangle$$

$$= \tilde{\beta}_k^* \tilde{\beta}_k \langle 0 | \cancel{a_{-k}^+ a_{-k}} + 1 | 0 \rangle$$

(using infrared regularization we have $[\tilde{a}_k, \tilde{a}_{k'}^+] = \delta_{k,k'}$)

$$= \tilde{\beta}_k^* \tilde{\beta}_k$$

Total particle number:

- The expected total number of particles at time η_2 is then:

$$\bar{N} = \sum_k \langle \Omega | \hat{N}_k | \Omega \rangle = \sum_k \tilde{\beta}_k^* \tilde{\beta}_k$$

- Note:
  - * We assumed here an infrared, i.e., a box regularization. (Else the number of created particles can only be 0 or ∞) ← Exercise: Why?
  - * Else, $\bar{N}$ may come out infinite, but that can be ok.
  - * This happens even for photon creation through moving charges.
  - * But we always must have of course finite "energy":

$$\langle \Omega | \hat{H}(\eta) | \Omega \rangle < \infty$$

Identification of the vacuum state

How can we identify, at any arbitrary fixed time, η, that Hilbert space vector, say $|\text{vacuum at } \eta\rangle$, which describes the vacuum, i.e., the no particle state, at that time, η?

Q: Is $|\text{vacuum at } \eta\rangle$ one of the (infinitely many) states

$$|0\rangle, |\tilde{0}\rangle, |\tilde{\tilde{0}}\rangle, \ldots$$

that come with choices of mode functions

$$v_k, \tilde{v}_k, \tilde{\tilde{v}}_k, \ldots$$

through $\quad a_k|0\rangle = 0, \; \tilde{a}_k|\tilde{0}\rangle = 0, \; \tilde{\tilde{a}}|\tilde{\tilde{0}}\rangle = 0, \ldots$?

A: As we will see:

Yes, if or when $|\text{vacuum at } \eta\rangle$ exists at all,

then there exist suitable mode functions, v_k,

(namely exactly one, up to a phase, for each k)

so that with

$$\hat{x}_k = \frac{1}{\sqrt{2}}\left(v_k^* a_k + v_k a_{-k}^+\right)$$

the state $|0\rangle$ defined through $a_k|0\rangle = 0$

is the vacuum state at the time η:

$$\boxed{|\text{vacuum at } \eta\rangle = |0\rangle}$$

But how to specify $|\text{vacuum at } \eta\rangle$?

We notice: To specify $|\text{vacuum at } \eta\rangle$ by specifying a suitable vector $|0\rangle$

is equivalent to

specifying a suitable mode function v_k (i.e. a suitable solution to the K.G. and Wronskian equations)

is equivalent to

specifying at time η that $v_k(\eta) = r_k$, $v_k'(\eta) = s_k$ for a suitable choice of $r_k, s_k \in \mathbb{C}$.

(because with the KG equation being 2^{nd} order in time, these two conditions suffice to determine the full v_k at all time.)

1st attempt:

- **Ansatz:**

  Let us try to define the vacuum state at a time η as that Hilbert space vector (up to a phase) which at time η minimizes the Hamiltonian, $H^{(x)}(\eta)$.

- To this end, we will choose $r_k, s_k \in \mathbb{C}$ suitably, so that $v_k(\eta) = r_k$, $v_k'(\eta) = s_k$ define that mode function v_k so that its $|0\rangle$ is the lowest energy state.

Calculation of the lowest energy state at some arbitrary fixed time, η_1.

$$\langle 0|\, \hat{H}^{(\chi)}(\eta_1)\, |0\rangle = \langle 0|\, \frac{1}{2}\int_{box} \hat{\chi}'^{\,2}(\eta_1,x) + \sum_{i=1}^{3} \hat{\chi}_{,i}^{\,2}(\eta_1,x)$$

$$+ \left(m^2 a^2(\eta_1) - \frac{a''(\eta_1)}{a(\eta_1)}\right) \hat{\chi}^2(\eta_1,x)\, d^3x\, |0\rangle$$

Exercise:

Use Fourier and use

$$\hat{\chi}_k(\eta_1) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta_1)\, a_k + v_k(\eta_1)\, a_{-k}^{+}\right)$$

to evaluate this energy expectation value.

Result:

$$\langle 0|\, \hat{H}^{(\chi)}(\eta_1)\, |0\rangle = \langle 0|\, \frac{1}{4}\sum_k \left(v_k'^{\,2}(\eta_1) + \omega_k^2(\eta_1)\, v_k^2(\eta_1)\right) a_k^+ a_{-k}^+$$

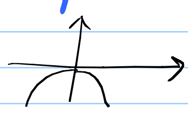
$$+ \frac{1}{4}\sum_k \left(v_k'^{*2}(\eta_1) + \omega_k^2(\eta_1)\, v_k^{*2}(\eta_1)\right) a_k\, a_{-k}$$

$$+ \frac{1}{2}\sum_k \left(|v_k'(\eta_1)|^2 + \omega_k^2(\eta_1)\, |v_k(\eta_1)|^2\right)\left(a_k^+ a_k + \frac{1}{2}\right) |0\rangle$$

$$= \frac{1}{4}\sum_k |v_k'(\eta_1)|^2 + \omega_k^2(\eta_1)\, |v_k(\eta_1)|^2$$

Here: the time-dependent frequency reads: $\omega_k^2(\eta) := k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$

Note: We assume $\omega_k^2(\eta) > 0$ because, else, the potential is inverted and there is no lowest energy state:



Recall:

* We defined $r_k := v_k(\eta_1)$, $s_k := v_k'(\eta_1)$
* We need to determine $r_k, s_k \in \mathbb{C}$
* This will determine a full mode function v_k with its a_k
* This determines a corresponding $|0\rangle$ obeying $a_k|0\rangle = 0$
* Our ansatz is then that:

$$|\text{vacuum at } \eta_1\rangle = |0\rangle$$

Concretely:

* From above, the energy at η_1 is:

$$\langle 0|\hat{H}^{(x)}(\eta_1)|0\rangle = \frac{1}{4}\sum_k |v_k'(\eta_1)|^2 + \omega_k^2(\eta_1)|v_k(\eta_1)|^2$$

* Using the definitions $r_k = v_k(\eta_1)$, $s_k = v_k'(\eta_1)$:

$$\langle 0|\hat{H}^{(x)}(\eta_1)|0\rangle = \frac{1}{4}\sum_k s_k s_k^* + \omega_k^2(\eta_1)\, r_k r_k^* \qquad (E)$$

* We want to minimize this expression, subject to the Wronskian condition

$$v_k'(\eta_1)\, v_k^*(\eta_1) - v_k(\eta_1)\, v_k^*(\eta_1)' = 2i$$

i.e., subject to the constraint:

$$s_k r_k^* - r_k s_k^* = 2i \qquad (C)$$

* Use Lagrange multiplier λ and extremize

$$S'(s_k, r_k) := s_k s_k^* + \omega_k^2 r_k r_k^* + \lambda(s_k r_k^* - r_k s_k^*)$$

* We have to solve:

$$\frac{\partial S}{\partial s_k^*}=0 \quad \text{i.e.,} \quad s_k - \lambda r_k = 0$$

$$\frac{\partial S}{\partial r_k^*}=0 \quad \text{i.e.,} \quad \omega_k^2 r_k + \lambda s_k = 0$$

along with the constraint (C): $s_k r_k^* - r_k s_k^* = 2i$

* Exercise:

Show that the solution is:

$$r_k = \frac{1}{\sqrt{\omega_k}} e^{i\theta} \qquad\qquad s_k = i\sqrt{\omega_k}\, e^{i\theta}$$

where $\theta \in [0, 2\pi)$ is arbitrary. We'll choose $\theta = 0$.

⇒ These conditions at time η_1,

$$\boxed{v_k(\eta_1) = \frac{1}{\sqrt{\omega_k(\eta_1)}} \quad , \quad v_k'(\eta_1) = i\sqrt{\omega_k(\eta_1)}}$$

define a mode function v_k for all η so that

$$\hat{x}_k(\eta_1) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta_1) a_k + v_k(\eta_1) a_{-k}^+\right)$$

and the corresponding state $|0\rangle$ obeying $a_k|0\rangle = 0$

is the lowest energy state of the Hamiltonian $\hat{H}^{(x)}(\eta_1)$,

i.e., the instantaneous lowest energy state at time η_1.

Special case: Minkowski space

- Minkowski space is the special case $a(\eta) = 1$ for all η.

  Then, $\omega_k^2(\eta) = \vec{k}^2 + m^2$ is a constant. Also: $\eta = t$.

- We conclude that $|0\rangle$ is the state of lowest energy at a time η_1 if we choose the mode functions which obey these conditions:

$$v_k(\eta_1) = \frac{1}{\sqrt{\omega_k}} \quad , \quad v_k'(\eta_1) = i\sqrt{\omega_k}$$

- Solving the K.G. eqn, we find that these mode fctns are:

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i(\eta-\eta_1)\omega_k} = \frac{1}{\sqrt{\omega_k}} e^{i(t-t_1)\omega_k}$$

Exercise:

* Verify that the state $|0\rangle$ that we have found for Minkowski space agrees with the state that we identified as the Minkowski space vacuum at the beginning of the course.

* Show that, if we, similarly, determine the lowest energy state at another time, η_2, then we obtain the same mode function v_k (up to an irrelevant phase).

* This means that the same vector $|0\rangle$ minimizes the energy at all times, on Minkowski space, (which had to come out because of time translation symmetry).

Back to our ansatz, namely the assumption:

At an arbitrary time η_1, the vacuum (no particle) state is that state which is the lowest energy state $|0\rangle$ at time η_1:

$$|\text{vacuum at } \eta_1\rangle = |0\rangle$$

- Implied prediction:

  Universe expands $\Rightarrow \hat{H}^{(x)}(\eta_1) \neq \hat{H}^{(x)}(\eta_2)$

  $\Rightarrow$ expect particle production, in general.

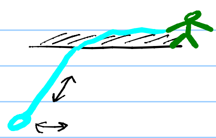
- Concretely: current production rate $\approx 10 \dfrac{\text{particles}}{(\text{km})^3\,\text{year}\cdot\text{species}}$!

  (and much higher in the faster-expanding early universe)

Experiment: That's much too high! We only have $\approx 10^9 \dfrac{\text{particles}}{(\text{km})^3}$ (particles with mass.)

Reconsider:

- Recall that any quantum system does not get excited (or only very little), if we change its parameters (e.g. the $\omega_k(\eta)$) "slowly".
- For the oscillator, "slow", is slow compared to the natural frequency of the oscillator.



Only changes in length which occur fast compared to the oscillator's frequency can parametrically excite the oscillator.

- Since the universe presently expands slowly, we should expect essentially no particle production, and indeed we don't see any, experimentally.

$\rightsquigarrow$ How to improve our ansatz for vacuum identification?

Preliminary consideration

- Consider models where the universe is initially Minkowski and then undergoes an expansion whose parameter change (of $\omega_k(\eta)$) is slow, i.e., adiabatic.

  Note: the overall change may still be large!

$\Rightarrow$ We expect essentially no particle creation.

$\Rightarrow$ The vacuum state (i.e. no particle state) should always be essentially the same Hilbert space vector.

$\Rightarrow$ Since there is only one vacuum state, $|0\rangle$, for all time, there is one mode function, v_k, whose $|0\rangle$ is the vacuum at all time.

How can we find this mode function v_k?

- Easy: We know $v_k(\eta)$ at very early times, when the universe was still Minkowski:

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k(\eta-\eta_0)}$$

arbitrary reference time

Then: the K.G. eqn. yields $v_k(\eta)$ at all time!

- Proposotion:

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k(\eta)}} e^{i\int_{\eta_0}^{\eta} \omega_k(\eta')d\eta'} \qquad (S)$$

is a very good approximation, if the evolution is "adiabatic".

Definition:

We say that a mode k evolves adiabatically slow, if:

Intuition:
$\frac{\omega'}{\omega^2}$ and $\frac{\omega''}{\omega^3}$ are rate of change of frequency compared to the frequency, and also rate of acceleration of frequency compared to the frequency.

$$\frac{\omega_k'(\eta)}{\omega_k^2(\eta)} \ll 1 \quad \text{and} \quad \frac{\omega_k''(\eta)}{\omega_k^3(\eta)} \ll 1 \qquad (AC)$$

Note: The denominators are chosen so that the quotients are unitless, because only pure numbers can reasonably be said to be small or large.

Exercise:

Prove the proposition.

Hint: Show that (S) obeys the K.G. eqn provided the adiabaticity, (AC), holds.

Is initial Minkowski period really necessary?

* Try to identify the v_k whose $|0\rangle$ is the adiabatically defined vacuum without referring to what v_k would look like in an earlier Minkowski period of the universe.

* Namely, try to identify v_k by a characteristic property that it has at all time.

* Indeed, we notice: (Exercise: check this)

Our v_k of (S) above satisfies at all times:

$$v_k(\eta) = e^{i\theta}\frac{1}{\sqrt{\omega_k(\eta)}}, \quad v_k'(\eta) = \left(i\omega_k(\eta) - \frac{1}{2}\frac{\omega_k'(\eta)}{\omega_k(\eta)}\right)\frac{e^{i\theta}}{\sqrt{\omega_k(\eta)}} \qquad (AV)$$

"The general adiabatic vacuum identification"

Definition:

* Consider an arbitrary time η_1.
* Assume that the evolution of ω_k is adiabatically slow for mode k, at time η_1.
* We then identify that state as the vacuum $|0\rangle$ (i.e. as the no particle state) at η_1 whose mode function v_k is specified by the conditions **(AV)** at η_1:

$$\boxed{v_k(\eta_1) = e^{i\theta}\frac{1}{\sqrt{\omega_k(\eta_1)}}\,,\quad v_k'(\eta_1) = \left(i\omega_k(\eta_1) - \frac{1}{2}\frac{\omega_k'(\eta_1)}{\omega_k(\eta_1)}\right)\frac{e^{i\theta}}{\sqrt{\omega_k(\eta_1)}}} \qquad \text{(AV)}$$

* We call this $|0\rangle$ the "adiabatic vacuum" at η_1.

Remarks:

- Recall that the criteria for chosing v_k so that its $|0\rangle$ is the lowest energy vacuum at time η_1 are:

$$v_k(\eta_1) = \frac{1}{\sqrt{\omega_k(\eta_1)}}\,e^{i\theta}\,,\qquad v_k'(\eta_1) = i\sqrt{\omega_k(\eta_1)}\,e^{i\theta} \qquad \text{(EV)}$$

- Note that AV and EV generally differ!

  $\Rightarrow$ The adiabatically-defined vacuum is generally not the lowest energy state!

- Note that the adiabatic vacuum criterion should only be applied when the evolution of the mode under consideration is actually adiabatic.
- No vacuum criterion for generic spacetimes is known.