QFT for Cosmology, Achim Kempf, Lecture 18

Note Title

Time evolution and the fluctuation spectrum:

Recall:

- We assume the system is in the state $|0\rangle$ which is the vacuum at η_0.
  $\Rightarrow$ The system is always in the state $|0\rangle$ (Heisenberg picture).
- We solve the QFT with $\hat{\chi}_k(\eta) := a(\eta)\,\hat{\phi}_k(\eta)$ and the ansatz

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\,a_k + v_k(\eta)\,a_{-k}^+\right)$$

where for convenience we choose the mode functions $\{v_k(\eta)\}_k$ so that $a_k|0\rangle = 0$.

- The technical challenge will be:
  - Identify $|0\rangle$, i.e., identify the initial conditions for the v_k at η_0.
  - Solve the K.G. eqn for the $v_k(\eta)$.

Benefit:

- State $|0\rangle$ known
- Operators $\hat{\phi}_k(\eta)$ known $\forall\, \eta > \eta_0$

$\Rightarrow$ We can calculate all predictions for all times, even, e.g., at times of nonadiabaticity or inverted potential!

In particular, we can calculate for all $\eta > \eta_0$:

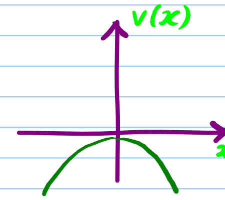
$$\boxed{\delta\phi_k(\eta) = k^{3/2}\left|\frac{v_k(\eta)}{a(\eta)}\right|}$$

We observe: The dynamics of $|v_k(\eta)|$ crucially affects $\delta\phi_k(\eta)$.

Q: In which circumstance does $v_k(\eta)$ grow most?

Answer: The most efficient mechanism to enlarge v_k occurs when the mode is nonadiabatically evolving in the sense that the mode oscillator is inverted:

$$\hat{\chi}_k''(\eta) + \overbrace{\omega_k^2(\eta)}^{\text{if } <0}\,\hat{\chi}_k(\eta) = 0$$

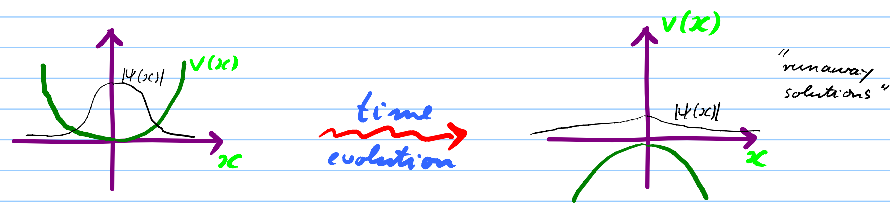


In such a time period, the Klein Gordon equation's solutions are not oscillatory because $\omega_k(\eta)$ is imaginary:

$$\omega_k(\eta) = \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$$

This term may be large enough to make the discriminant negative.

* Instead, there will be one exponentially decaying and one exponentially growing solution. Inverting a harmonic oscillator is an efficient way to increase $\Delta\phi$:



□ **Caveat:**

Notice that this argument applies to x but $\phi = \frac{1}{a} x$.
Thus, ϕ's growth is slower than that of x.
Recall: The equation of motion of ϕ has a friction-type term.

Before we calculate the fluctuation amplification explicitly:

Relationship of fluctuation amplification to particle creation

- Assume that at a later time, η_1, the evolution is adiabatic for mode k (i.e. its ω_k changes slowly).

  $\Rightarrow$ We can identify $|vac_{\eta_1}\rangle$:

  Using the adiabatic vacuum identification criterion, we find the mode function $\tilde{v}_k$ for which:

$$|\tilde{0}\rangle = |vac_{\eta_1}\rangle$$

- Case 1: The evolution of mode k was adiabatic from η_0 to η_1.

  * Therefore:

$$v_k = \tilde{v}_k \quad \text{and} \quad |0\rangle = |\tilde{0}\rangle$$

  * Therefore:

  The state of the system, $|\Omega\rangle = |0\rangle$, is still the vacuum state at time η_1:

$$|\Omega\rangle = |\tilde{0}\rangle$$

  * There is no particle creation.

  * But since $v_k = \frac{1}{\sqrt{\omega_k(\eta)}} e^{i\int_{\eta_0}^{\eta_1} \omega(\eta')d\eta'}$, in general:

$$|v_k(\eta_1)| \neq |v_k(\eta_0)| \qquad \left(\text{namely } |v_k(\eta)| = \omega_k(\eta)^{-\frac{1}{2}}\right)$$

  $\Rightarrow$ the fluctuations, which depend on $|v_k(\eta)|$ can be affected even if there is no particle creation.

II **Case 2**: The evolution was not always adiabatic between η_0 and η_1.

* Then,

$$v_k \neq \tilde{v}_k$$

* But since both are in the same 2 dimensional solution space to the K.G. equation, there exist α_k, β_k:

Recall: When particle concept applies, $|\beta_k|$ yields nonadiabatic particle production

$$v_k(\eta) = \alpha_k \, \tilde{v}_k(\eta) + \beta_k^* \, \tilde{v}_k^*(\eta)$$

* Substitute in the fluctuations equation:

$$\delta\phi_k(\eta)^2 = a^{-2}(\eta)\, k^3 \, |v_k(\eta)|^2$$

$$= a^{-2}(\eta)\, k^3 \, |\alpha_k \tilde{v}_k(\eta_1) + \beta_k^* \tilde{v}_k^*(\eta_1)|^2$$

* For clarity, assume that the nonadiabatic period is over by η_1.

* Also, assume that spacetime is again Minkowski around η_1. (Thus, we focus on nonadiabatic effects only)

* In this case:

$$\tilde{v}_k(\eta) = \frac{1}{\sqrt{\omega_k(\eta_1)}} e^{i\omega_k(\eta_1)\eta} \quad \text{for all } \eta \approx \eta_1$$

$$\Rightarrow \delta\phi_k^2(\eta) = a^{-2}(\eta)\frac{k^3}{\omega_k(\eta_1)}\left(|\alpha_k|^2 + |\beta_k|^2 - 2\,Re\big(\underbrace{\alpha_k \beta_k e^{2i\omega_k(\eta_1)\eta}}_{\text{over a long enough time period this term averages 0.}}\big)\right)$$

* We use: $|\alpha_k|^2 - |\beta_k|^2 = 1$ (W)

$$\Rightarrow \quad \delta\phi_k^2(\eta_1) = \underbrace{a^{-2}(\eta_1)\frac{k^3}{\sqrt{k^2+m^2a^2(\eta_1)}}}_{\text{This term is the same, whether or not the expansion of spacetime has been adiabatic.}}\Big(1 + 2\underbrace{|\beta_k|^2}_{\text{This term is only non zero if the evolution was non-adiabatic.}}\Big) \qquad (F)$$

* **Notice**: $|\beta_k|$ and $|\alpha_k|$ can both become very large. This is consistent with the Wronskian condition, (W).

* **Particle production:**

Recall that the expected number of created particles is also given by $|\beta_k|^2$:

$$\bar{\tilde{N}}_k(\eta_1) = \langle \Omega | \hat{\tilde{N}}_k | \Omega \rangle = \ldots$$

$$= |\beta_k|^2$$

$\tilde{N}_k = \tilde{a}_k^+ \tilde{a}_k$

* **Remark:**

But (F) holds even if $\omega_k^2 < 0$ at η_1! We know what we mean by field fluctuations even when we do not have a concept of vacuum and particles.

$\Rightarrow$ 'Quantum fields more fundamental than quantum particles.'

Fluctuations in proper coordinates as opposed to comoving coordinates

* We have: $\underset{\substack{\uparrow\\ \text{proper length}}}{d} = a(\eta)\underset{\substack{\uparrow\\ \text{comoving length}}}{L}$, $\quad \underset{\substack{\uparrow\\ \text{proper momentum}}}{p} = \frac{1}{a(\eta)}\underset{\substack{\uparrow\\ \text{comoving momentum}}}{k}$

* Therefore, $\delta\phi_k^2(\eta) = a^{-2}(\eta)\frac{k^3}{\sqrt{k^2+m^2a^2(\eta)}}\left(1+2|\beta_k|^2\right)$ becomes:

$$\delta\phi_p^2(\eta) = a^{-2}\frac{a^3p^3}{\sqrt{a^2p^2+a^2m^2}}\left(1+2|\beta_{ap}|^2\right)$$

$$= \underbrace{\frac{p^3}{\sqrt{p^2+m^2}}}_{\text{same as Minkowski}}\left(1+2|\beta_{ap}|^2\right)$$

* **Note:** The nonadiabatic term depends on p.

* **Note:** This was the case where we end in a Minkowski space. We see that in this case we must get back the original Minkowski spectrum if the evolution from η_0 to η_1 was adiabatic.

* **Note:** Also in thermodynamics, "adiabatic" processes are reversible processes.

Application to specific cosmological models

The standard model of cosmology holds that the very early universe underwent a short period of almost exponential expansion, "inflation".

→ Begin by studying QFT in de Sitter spacetime:

The de Sitter FRW spacetime can be defined through

$$\boxed{a(t) := e^{Ht} \quad \text{for all } t \in \mathbb{R}}$$

Notes: * t is the time on a comoving observer's wrist watch
* large $H \Leftrightarrow$ large acceleration

Here: $H > 0$ is a constant, the "Hubble constant".

☐ Exercise: Read Mukhanov's comments on de Sitter space.

The de Sitter horizon

Proposition: (in particle picture) (Note: large $H \Leftrightarrow$ small horizon d_H)

Objects (or any observers) who are further apart than a proper distance of $d_H = 1/H$ can never meet, and cannot communicate.

Proof:

* Consider an observer in a galaxy **A**. Let us choose the origin of the comoving coordinate system(s) to be where this observer sits.
* Now suppose that, at some arbitrary time, t_s, this observer sends a radio signal towards another galaxy, B.
* The signal travels in a small time Δt the small comoving distance Δx:

proper distance

our unit convention here

$$\frac{a(t)\,\Delta x}{\Delta t} = c = 1$$

speed of light

$$\Rightarrow \quad \frac{dx}{dt} = a^{-1}(t) \quad \text{i.e.:} \quad \frac{dx}{dt} = e^{-Ht}$$

$$\Rightarrow \quad x(t) = -\frac{1}{H}e^{-Ht} + C_1$$

Fix the integration constant C_1 so that $x(t_s) = 0 \Rightarrow C_1 = \frac{1}{H}e^{-Ht_s}$

(Recall: The proper distance traveled is: $d(t) = a(t)\,x(t)$. Clearly: $d(t) \to \infty$ as $t \to \infty$)

$$\Rightarrow \quad x(t) = -\frac{1}{H}e^{-Ht} + \frac{e^{-Ht_s}}{H}$$

$\Rightarrow$ As $t \to \infty$ we have $x(t) \to \frac{e^{-Ht_s}}{H}$.

Terminal comoving distance traveled.

Thus, can reach galaxy B if comoving distance is at most $d_c = \frac{e^{-Ht_s}}{H}$.

Q: Proper distance d_p of such B from A at t_s?

A: $d_s = a(t)\,d_c \Rightarrow d_s = e^{+Ht_s}\frac{e^{-Ht_s}}{H} = \frac{1}{H}$

Recall: This holds for arbitrary t_s.

$\Rightarrow$ A signal sent by A at any time t_s can only ever reach B if at the time of sending, t_s, the proper distance between A and B is at most $\frac{1}{H}$.

$\Rightarrow$ Any two observers further apart than a proper distance of $1/H$ cannot communicate!

Interpretation: In the case where a de Sitter exponential expansion lasts forever, between any objects of proper distance $> 1/H$, space is being created faster than what can be crossed when travelling with the speed of light.

(Remark: Notice that the proper size of the de Sitter horizon is constant in time.)

Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper wavelength obeys $\lambda \ll \frac{1}{H}$ but stop oscillating and possess instead an imaginary frequency when their proper wavelength has grown beyond, i.e., when $\lambda \gg \frac{1}{H}$, assuming that their mass is small: $m \ll H$.

Proof: 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

Recall: $\eta(t) := \int^t \frac{1}{a(t')} dt'$

here: $\eta(t) = \int^t e^{-Ht'} dt'$

$= -\frac{1}{H} e^{-Ht} + C_1$

The choices of the integration constant C_1 merely mean different fixed shifts in the time coordinate η relative to the time coordinate t.

Notice:

As $t \to -\infty$ we have $\eta \to -\infty$.

But as $t \to +\infty$ we have $\eta \to C_1'$.

Choose $C_1' = 0$:

$\Rightarrow$

$$\eta(t) = -\frac{1}{H}\frac{1}{a(t)}$$

$$a(t) = -\frac{1}{H\eta(t)}$$

i.e.:

$$\boxed{a(\eta) = -\frac{1}{H\eta}}$$

2) Introduce $\hat{\chi}_k(\eta) := a(\eta)\,\hat{\phi}_k(\eta)$:

We have: $\hat{\chi}_k(\eta) = -\frac{1}{H\eta}\hat{\phi}_k(\eta)$

$\hat{\chi}_k$ obeys this Klein Gordon equation

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta)\,\hat{\chi}_k(\eta) = 0$$

with:

$$\omega_k^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

Exercise: Show that in the de Sitter case this yields:

$$\boxed{\omega_k^2(\eta) = k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}$$

3.) Check for imaginary frequencies.

We are assuming $m \ll H$. $\Longrightarrow \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} < 0$

- Thus, in $\omega_k^2(\eta) = k^2 + \underbrace{\frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}_{\text{we have: } <0}$

- Therefore: For each mode k there comes a time when ω_k^2 becomes negative!

The case relevant in cosmology: $m = 0$ (we'll assume this)

$\Longrightarrow$ The time when a mode k crosses the horizon is given by:

$$\boxed{\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k}}$$

4.) Conclusion:

Recall: $\eta \in (-\infty, 0)$ i.e. $|\eta| \gg 1/k$ means early times.

- A mode oscillates as long as:

$$|\eta| \gg \frac{1}{k} \quad \text{i.e., while } |\eta| k \gg 1 \qquad (a)$$

(Used that $\sqrt{2}$ and 1 are of same order of magnitude)

- A mode has imaginary frequency from when

This is late times, i.e. when $\eta \approx 0$.

$$|\eta| \ll \frac{1}{k} \quad \text{i.e., from when } |\eta| k \ll 1 \qquad (b)$$

Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = 2\pi/L$ we obtain:

L — comoving wavelength

$$|\eta| k = \frac{1}{Ha} \frac{2\pi}{L}$$

Transforming to the proper wavelength, $\lambda = a(\eta)L$, we obtain:

$$|\eta| k = \frac{2\pi}{H\lambda}$$

(Thus, the proper wavelength, λ, of a fixed comoving mode, k, obeys: $\lambda(\eta) = \frac{2\pi}{Hk|\eta|}$)

Thus, finally, the two cases, (a) and (b) become:

- A mode oscillates as long as: $|\eta| k \gg 1$

  i.e., as long as $\frac{2\pi}{H\lambda} \gg 1$ i.e.: $\boxed{\lambda \ll \frac{1}{H}}$ (a')

- A mode has imaginary frequency from when:

  $\frac{2\pi}{H\lambda} = |\eta| k \ll 1$, i.e., from when $\boxed{\lambda \gg \frac{1}{H}}$ (b')

This is what we had set out to show.