QFT for Cosmology, Achim Kempf, Lecture 19

Note Title

Recall de Sitter model spacetime:

$$a(t) := e^{Ht}$$

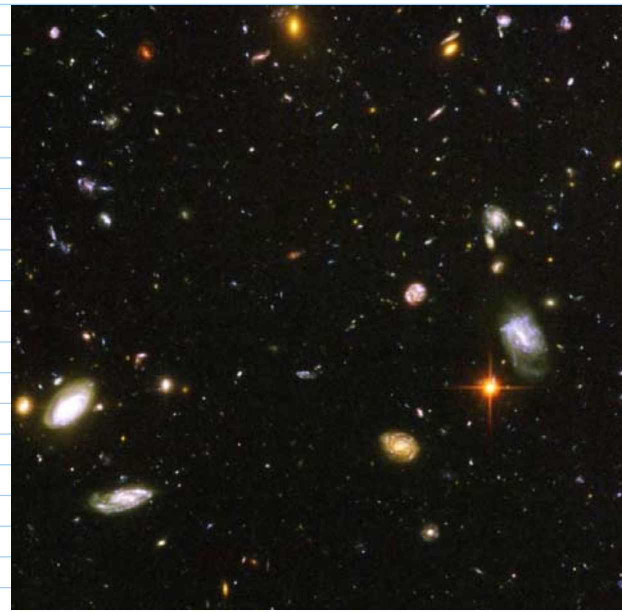
$$\eta(t) = -\frac{1}{H} e^{-Ht}$$

$$a(\eta) = -\frac{1}{H\eta}$$

$$\hat{x}_k''(\eta) + \omega_k^2(\eta)\,\hat{x}_k(\eta) = 0$$

$$\omega_k^2(\eta) = k^2 - \frac{2}{\eta^2}$$

Note: Here, we neglect the mass term



$\Rightarrow$ For a mode $k$ the sign flip of $\omega_k^2(\eta)$ occurs at the time:

$$\boxed{|\eta_c(k)| = \sqrt{2}/k}$$

This is also roughly the time when its proper wavelength

$$\lambda_k(\eta) = \frac{2\pi}{k} a(\eta) = \frac{2\pi}{k} \frac{1}{H|\eta|}$$

reaches the size of the Hubble horizon $d_H = 1/H$:

Check: $\lambda_k(\eta) = d_H$

is $\frac{2\pi}{k} \frac{1}{H|\eta|} = \frac{1}{H}$

$$|\eta| = \frac{2\pi}{k} \approx \frac{\sqrt{2}}{k} \quad \checkmark$$

## The more realistic case of a de Sitter expansion of a finite duration

- Consider the case that spacetime was exponentially expanding only in a finite time interval:

$$\eta_i < \eta < \eta_f$$

and assume that spacetime was expanding slowly or was even Minkowski before $\eta_i$, and after $\eta_f$.

- **Recall:** The time when a mode, $k$, crosses the horizon is:

$$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k}$$

$\Longrightarrow$ modes $k$ with $\eta_{hor}(k) \notin [\eta_i, \eta_f]$ never cross the de Sitter horizon!

## $\Rightarrow$ Three classes of modes:

### 1. "Small" modes:

By the time their proper wavelength would reach the Hubble horizon length the de Sitter period is already over:

$$\eta_{hor}(k) >> \eta_f \qquad \text{Recall: Both sides are negative}$$

i.e.:

$$\frac{\sqrt{2}}{k} \ll |\eta_f| \qquad \text{Recall: } \eta_{hor} \approx \frac{-\sqrt{2}}{k}$$

$$L \ll |\eta_f|$$

Their quantum fluctuations do not get "amplified", as we will see.

## 2. "Medium" size modes:

These are the modes which do cross the horizon because

$$\eta_i < \eta_{hor}(k) < \eta_f$$

The quantum fluctuations of those modes are important in cosmology.

## 3. "Large" modes:

These are modes which were larger than the horizon already at $\eta_i$. In the inflationary scenario they are today very much larger than the visible universe. They may only contribute, effectively, like a cosmological constant - and may even be the origin of $\Lambda$.

## Quantum fluctuations in de Sitter space.

- The usual ansatz

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a^+_{-k}\right)$$

succeeds, as always, for any function $v_k$ which obeys:

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0 \qquad \text{(a)}$$

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k'^*(\eta) = 2i \qquad \text{(b)}$$

- The solution space of (a) can be shown to be spanned, for example, by these two real-valued Bessel functions

$$u_k(\eta) := \sqrt{k|\eta|}\; J_n(k|\eta|)$$

(not complex conjugation, just another symbol)

$$\bar{u}_k(\eta) := \sqrt{k|\eta|}\; Y_n(k|\eta|)$$

generalizations of sine and cosine

where:

$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

Thus: every mode function $v_k$ is a linear combination

$$v_k(\eta) = A_k\, u_k(\eta) + B_k\, \bar{u}_k(\eta) \qquad (*)$$

with complex coefficients $A_k, B_k$.

## How to identify the state of the system?

Strategy:

a.) Check if modes start out in an adiabatic regime (the small and medium ones do).

b.) Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then.

c.) Choose mode function $v_k$ whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

Then: d.) Calculate $\delta\phi_k$ at the end of the exponential expansion, $\eta_f$, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f)\, k^3\, |v_k(\eta_f)|^2$$

**Important:** We know that $v_k$ is a linear combination of $u_k$ and $\bar{u}_k$ and we know $u_k$ and $\bar{u}_k$ explicitly. Thus, we only need to find $A_k$ and $B_k$ in $(*)$ !

a.) Check if modes start out in an adiabatic regime.

Indeed, we see from the K.G. eqn.

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

that at very early times, $\eta \ll 0$, we have roughly Minkowski:

$$v_k''(\eta) + k^2 v_k(\eta) = 0$$

(except if $k$ is very small, i.e., for very large modes.)

b.) Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then – namely the Minkowski vacuum.

Note: we could also use the adiabatic vacuum criterion, with little difference.

c.) Choose mode function $v_k$ whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

Thus, $v_k$ is the usual Minkowski mode function at early times:

$$v_k = \frac{1}{\sqrt{\omega_k}}\, e^{i\omega_k \eta + i\alpha} \quad \text{for} \quad \eta \ll 0$$

(we are neglecting the mass term for simplicity, and because it is realistic) $\rightarrow$ i.e.

$$v_k = \frac{1}{\sqrt{k}}\, e^{ik\eta + i\alpha} \quad \text{for} \quad \eta \ll 0$$

**Technical observation:** At early times, $\eta \ll 0$:

$$u_k(\eta) \approx \sqrt{\frac{2}{\pi}} \cos(k|\eta| + \text{const})$$

same constant

$$\bar{u}_k(\eta) \approx \sqrt{\frac{2}{\pi}} \sin(k|\eta| + \text{const})$$

$\Rightarrow$ **Proposition:**

In terms of $u_k$, $\bar{u}_k$ the mode function $v_k$ reads:

$$v_k(\eta) = \overbrace{\sqrt{\frac{\pi}{2k}}}^{A_k \,=} u_k(\eta) - i \overbrace{\sqrt{\frac{\pi}{2k}}}^{B_k \,=} \bar{u}_k(\eta)$$

i.e.:

$$\boxed{v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left( J_m(k|\eta|) - i\, Y_m(k|\eta|) \right)}$$

Proof: Exercise.

d.) Now we can calculate $\delta\phi_k$ at the end of the exponential expansion, $\eta_f$, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f)\, k^3 \, |v_k(\eta_f)|^2$$

**Case 1:** Very small modes

- They are those with $k$ large enough, so that in

$$v_k''(\eta) + \left( k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} \right) v_k(\eta) = 0$$

the $k^2$ term dominates all through the expansion.

- These modes never cross the horizon and we have, approximately:

I.e., the Bessel functions in the mode function stay sine and cosine in good approximation for all times $\eta$ up to $\eta_f$.

$$v_k(\eta) = \frac{1}{\sqrt{k}} e^{ik\eta} \quad \text{for all } \eta$$

- Thus:

The vacuum fluctuations at the end of the exponential expansion are still as in Minkowski case:

Recall:

$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} |v_k(\eta_f)|$

$$\delta\phi_\lambda(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} \frac{1}{\sqrt{k}}\Big|_{k\approx L^{-1}}$$
$$= \frac{1}{a(\eta_f) L}$$
$$= 1/\lambda(\eta_f)$$

proper wavelength at time $\eta_f$. (neglecting factors of $2\pi$)

- Recall: This is the usual fluctuation spectrum for massless fields in Minkowski space:

Fluctuations with large proper spatial extent $\lambda$ are still suppressed.

## Case 2: Medium size modes.

- They are those with $k$ so that in

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

the sign changes at a time $\eta_{hor}(k)$ during the exponential expansion:

$$\eta_i < \eta_{hor}(k) < \eta_f$$

Let us evaluate the fluctuation spectrum

$$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2}\, |v_k(\eta_f)|$$

at the time $\eta_f$, i.e., when the exponential expansion ends:

Then, the K.G. eqn. is to a good approximation:

Recall:

$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$

$$v_k''(\eta) - \frac{2}{\eta^2}\, v_k(\eta) = 0$$

and a basis of solutions is easy to find, e.g.:

$$w_k^{(1)}(\eta) = (k|\eta|)^2 \qquad \text{decaying for } \eta \to 0$$

$$w_k^{(2)}(\eta) = \frac{1}{k|\eta|} \qquad \text{growing for } \eta \to 0$$

Indeed: use this property of the Bessel functions:

Recall:

$u_k(\eta) := \sqrt{k|\eta|}\, J_n(k|\eta|)$

$\bar{u}_k(\eta) := \sqrt{k|\eta|}\, Y_n(k|\eta|)$

$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{3}{2}$

$$\left.\begin{array}{l} u_k(\eta) \to \frac{2^{-n}}{\Gamma(n+1)} (k|\eta|)^{n+\frac{1}{2}} \to 0 \\ \\ \bar{u}_k(\eta) \to \frac{-\Gamma(n)}{\pi} 2^n (k|\eta|)^{\frac{1}{2}-n} \to \infty \end{array}\right\} \text{as } \eta \to 0 \text{ (i.e. as } t \to \infty)$$

Therefore, for late $\eta$:

Recall:

$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}\left(J_n(k|\eta|) - i\, Y_n(k|\eta|)\right)$

$$v_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}}_{B_k}\, \frac{\Gamma(n)}{\pi}\, 2^n (k|\eta|)^{\frac{1}{2}-n} + \text{negligible}$$

Recall:

$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2}\, |v_k(\eta_f)|$

$$\delta\phi_L(\eta_f) \approx \left. \underbrace{H\eta_f}_{a^{-1}(\eta_f)}\, k^{3/2} \sqrt{\frac{\pi}{2k}}\, \frac{\Gamma(n)}{\pi}\, 2^n (k|\eta_f|)^{\frac{1}{2}-n} \right|_{k=L^{-1}}$$

$$\Rightarrow \quad \delta\phi_L(\eta_f) \approx H\left(\frac{|\eta|}{L}\right)^{\overbrace{\frac{3}{2}-n}^{\approx 0}} \cdot \Gamma(n)\,\frac{2^n}{\pi}$$

For comparison, recall case 1, small modes, whose fluctuation amplitudes are as on Minkowski space:
$\delta\phi_\lambda \approx 1/\lambda$

independent of $\eta_f$! $\Rightarrow$ May as well evaluate right after horizon crossing.

$$\boxed{\delta\phi_L(\eta_f) \approx \underbrace{H\cdot 2^{3/2}\Gamma(3/2)/\pi}_{\text{independent of } L \Rightarrow \text{ indep. also of } \lambda\,!}} \quad \text{for } n = 3/2$$

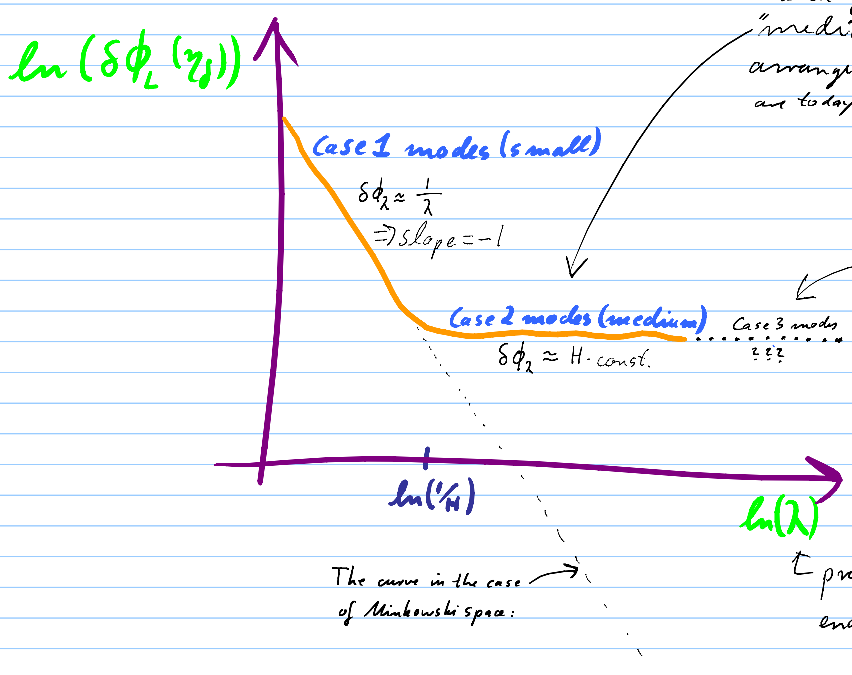
$\Rightarrow$ The medium sized modes get amplified just enough so that the usual suppression of fluctuations of large spatial extent is compensated.

$\Longrightarrow$ The quantum fluctuations of a comoving mode when its proper wavelength $\lambda$ is getting larger than the Hubble length, i.e., when $\lambda > \lambda_{Hubble} = 1/H$, remain as large in amplitude as they where when $\lambda = \lambda_{Hubble} = 1/H$

Indeed: $\delta\phi_L(\eta_f)$ does not depend on $\eta_f$: Fluctuations stay of same amplitude during de Sitter expansion. (even though their physical wavelength grows!)

$\Rightarrow$ After exponential expansion:

As we'll see, in a suitable model of very early universe cosmology, "medium size" can be arranged to mean modes that are today at cosmological scales.




unknown significance (depends on assumption about their initial conditions before the expansion, at $\eta_i$: there was no vacuum state for them!)

proper wavelength at $\eta_f$, the end of the exponential expansion

## Preliminary estimates:

* If this is the seeding mechanism for cosmic structure formation, then:

* H determines the amplitude of the later-observed fluctuations and must be of the right size to conform with observations. Measurements of the CMB indicate:

$$H^{-1} \approx 10^5 \, l_{planck} \approx 10^{-29} m$$

* The interval $[\eta_i, \eta_f]$ must be long enough so that such small modes have time to expand to cosmological size. For example this time period would do:

$$[10^{-34} s, \; 10^{-32} s]$$

$\Rightarrow$ how much expansion?

$$\frac{a(t_f)}{a(t_i)} = e^{H(t_f - t_i)c} = e^{\frac{10^{-32} s \cdot 3 \cdot 10^8 m}{10^{-29} m \cdot s}} = e^{3 \cdot 10^5}$$

## Realistic cosmic inflation

### 1. How can a period of near-exponential expansion be caused?

**Recall the full action:**

We neglect such terms by Occam's razor: there is no evidence for their existence as yet.

$$S = -\frac{1}{16\pi G} \int \Big[ 2\underbrace{\Lambda}_{\text{cosm. constant}} + R(x) + \cancel{\mathcal{O}(R\phi) + \mathcal{O}(R^2) + \ldots} \Big] \sqrt{|g|} \, d^4x$$

$$+ \int \left[ \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right] \sqrt{|g|} \, d^4x$$

Note: $\phi$ is now called the "Inflaton" field.

$$+ \cancel{S_{other\ fields}}$$

$\leftarrow$ We neglect this term because the contribution of the inflaton field $\phi$ and of $g_{\mu\nu}$ are assumed to have been dominant in the very early universe.

Example choice of $V$: $\quad V(\phi) = m\phi^2 + \lambda\phi^4$

## Equations of motion:

* $\dfrac{\delta S}{\delta \phi(x)} = 0$ yields the K.G. eqn.:

$$\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(x)\,\phi_{,\nu}(x)\sqrt{|g(x)|}\right) + \frac{\partial V}{\partial \phi}(x)\sqrt{|g(x)|} = 0 \qquad \text{(KG)}$$

* $\dfrac{\delta S}{\delta g_{\mu\nu}(x)} = 0$ yields the Einstein eqn.:

$$R_{\mu\nu}(x) - g_{\mu\nu}(x)R(x) + \Lambda g_{\mu\nu}(x) = -8\pi G\, T_{\mu\nu}(x) \qquad \text{(E)}$$

where the energy-momentum tensor (for $\phi$ only) reads:

$$T_{\mu\nu} = \phi_{,\mu}\,\phi_{,\nu} - g_{\mu\nu}\left(g^{\alpha\beta}\phi_{,\alpha}\,\phi_{,\beta} - V(\phi)\right) + \underbrace{\cancel{T_{\mu\nu}^{(\text{other fields})}}}$$

We'll assume this small compared to the contribution of $\phi$, during the very early universe.

## The important special case of homogeneity & isotropy

Eqns. (KG) and (E) are a set of coupled nonlinear partial differential equations which are even classically very hard.

⇝ As a lowest order approximation we assume perfect homogeneity & isotropy:

$$\phi(x,t) = \phi(t)$$

$$g_{\mu\nu}(x,t) = g_{\mu\nu}(t)$$

**Note:**

This may also be viewed as considering only the $k=0$ modes, neglecting all other modes.

Thus, the eqns of motion simplify:

$\left(\dot{\cdot} = \frac{\partial}{\partial t}\right)$ $$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0$$ (K.G. eqn.)

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\, T^0_0 + \Lambda$$ (the 0,0 component of the Einstein equation)

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} = 8\pi G\, T^i_i - \Lambda$$ (no sum) (the $i,i$ components of the Einstein equation)

Here: $T^0_0 = \rho(t) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ (the energy density $\rho$ of $\phi$)

$T^i_i = p(t) = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ (the pressure $p$ of $\phi$)

Given any initial conditions and given any $V(\phi)$ one can now solve for $a(t)$, $\phi(t)$, at least numerically!

First attempt to get exponential expansion:

Assume that $\Lambda$ dominates over $T_{\mu\nu}$ of all fields in nature.

Then, the 0,0 component of Einstein's equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} T^0_0 + \frac{1}{3}\Lambda \quad \text{becomes} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\Lambda$$

whose solution has the desired behavior:

$$a(t) = a_0 e^{Ht} \quad \text{with} \quad H = \sqrt{\Lambda/3} \ !$$

Problems:
- The measured $\Lambda$ is too tiny: $\Lambda_{obs} \approx 10^{-52} m^{-2}$
  We'd need a $\Lambda$ closer to the Planck scale $\Lambda_{Planck} \approx 10^{+70} m^{-2}$.
- Since $\Lambda$ is constant, such an inflation would never end.

Solution: A temporarily large $V(\phi)$ has same effect!

**Notice:**

- The cosmological constant $\Lambda$ contributes effectively a positive energy density $\rho_\Lambda$ and effectively a negative pressure $\rho_p$.

- Vice versa, whenever $V(\phi) \gg \dot{\phi}^2/2$ then $V(\phi)$ temporarily plays the same rôle as $\Lambda$.

- How close we are to $V(\phi) \gg \dot{\phi}^2/2$ is described by the "Equation of state parameter":

$$w(t) := \frac{p(t)}{\rho(t)} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} \qquad -1 < w < 1$$

$\Rightarrow$ Iff $w \approx -1$ then $V(\phi)$ acts like a cosm. constant.