Note Title

Perturbative quantization of inflaton field and the metric.

Recall:

- We decompose the inflaton field $\phi(x,\eta)$:

$$\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta)$$

where:

* $\phi_0(\eta)$ is assumed large and is treated classically.

* $\varphi(x,\eta) =: \delta\phi(x,\eta)$ describes a field of small inhomogeneities and is to be quantized: $\hat{\varphi}(x,\eta)$

- We decompose the metric $g_{\mu\nu}(x,\eta)$:

$$g_{\mu\nu}(x,\eta) = \underbrace{a_0^2(\eta)\,\eta_{\mu\nu}}_{\text{treated classically}} + \underbrace{\gamma_{\mu\nu}(x,\eta)}_{\text{assumed small, to be quantized}}$$

- Here, $\gamma_{\mu\nu}(x,\eta)$ can be decomposed into scalar, vector and tensor-type inhomogeneities, using functions $E, B, \Psi, \Phi, V_i, W_i, h_{ij}$.

namely: $ds^2 = g_{\mu\nu}(x,\eta)\,dx^\mu dx^\nu$

$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

$$ds_S^2 = a^2(\eta)\Big[2\Phi(x,\eta)\, d\eta^2 - 2\sum_{i=1}\frac{\partial}{\partial x^i}B(x,\eta)\, dx^i d\eta$$
$$-\sum_{i,j=1}^{3}\left(2\Psi(x,\eta)\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}E(x,\eta)\right)dx^i dx^j\Big]$$

$$ds_V^2 = a^2(\eta)\Big[2\sum_{i=1}^{3}V_i(x,\eta)\, dx^i d\eta$$
$$-\sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x^j}W_i(x,\eta) + \frac{\partial}{\partial x^i}W_j(x,\eta)\right)dx^i dx^j\Big]$$

$$ds_T^2 = a^2(\eta)\sum_{i,j=1}^{3}h_{ij}(x,\eta)\; dx^i dx^j$$

We insert the approximation

$$\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta)$$

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta)$$

with φ, γ assumed small, into the action:

$$S = \frac{-1}{16\pi G}\int R\sqrt{|g|}\, d^4x$$

$$+\frac{1}{2}\int(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\sqrt{|g|}\, d^4x$$

$$+ \text{neglected} \quad (\text{other fields})$$

One obtains many terms with $\Phi, \Psi, B, E, V, W, h$!

- These terms can be simplified! Why?

  Now that space is curved, there is no longer a preferred foliation of spacetime into spacelike hypersurfaces!

  ⇒ No preferred choice for the coordinate system.

  (e.g., no prefered conformal time & space cds)

- But the choice of cds will affect the functions above, i.e. they are in part coordinate system dependent.

  ⇒ We may choose our $\underbrace{\text{spacelike hypersurfaces}}_{\text{and thus our notion of equal time}}$ so that these functions $\Phi, \Psi, E, B, v, w, h$ vanish or simplify.

It took on the order of 10 years to clarify this "gauge" question!

- For detailed references, see e.g.:

  * A. Riotto, hep-ph/0210162 (relatively compact)

  * R. H. Brandenberger et al, Physics Reports 215, 203 (1992) (long)

- Result:

  * For small inhomogeneities (1st order perturbation) nearly all inhomogeneities can be eliminated by suitable coordinate choice.

  * Except, there are two fields, which are coordinate system, i.e., "gauge" independent. Namely:

I) A spatial tensor field:

This is $h_{ij}(x,\eta)$ itself. It represents $T_{\mu\nu}$-independent, so-called Weyl curvature, namely gravitational waves. $h_{ij}(x,\eta)$ measures how much space is locally distorted against itself in different directions.

II) A spatially scalar field, r, made of φ and $g_{\mu\nu}$'s scalar part:

Due to the Einstein eqn,

$$\delta\phi(x,\eta) = \varphi(x,t)$$

combines with the scalar part of the metric inhomogeneities

$$\Psi(x,\eta),$$

to yield one dynamical entity, namely:

recall: $\phi_0(\tau)$ = classical homogeneous inflaton field.

$$r(x,\eta) := -\frac{a_0'}{a_0}\left(\phi_0(\eta)'\right)^{-1}\underbrace{\varphi(x,\eta)}_{\text{from inflaton}} - \underbrace{\Psi(x,\eta)}_{\text{from "scalar" part of the metric}}$$

Physically, what is $r(x,\eta)$?

* First term: $\Psi(x,\eta)$ is the (scalar) metric's fluctuation.
* Second term: In $\frac{a'}{a}\frac{1}{\phi_0'}\varphi$, the $\varphi(x,\eta)$ is the scalar field's fluctuation

Consider now: 2 Useful choices for foliations of spacetime into spacelike hypersurfaces of equal time:

a.) Foliate so that on surfaces of equal time, η, one has: $\varphi \equiv 0$.

⇝ Equal time hypersurfaces chosen so that all points of equal value of ϕ have equal value of time.

Note: Only possible if ϕ decays over time (e.g. slow roll inflation, but not de Sitter).

⇝ We see that $r(x,\eta)$ expresses now purely metric fluctuations

⇝ Technically, these are fluctuations in the "intrinsic curvature". (Local bloating)

b.) Foliate so that on surfaces of equal time, η, one has: $\Psi \equiv 0$

In this case, along each equal time surface there is no local bloating - but instead the inflaton field fluctuates.

Recall:

$$r(x,\eta) := -\frac{a_0'}{a_0}\left(\phi_0(\eta)'\right)^{-1}\varphi(x,\eta) - \Psi(x,\eta)$$

Question:

Why does the contribution of the inflaton in $r(x,\eta)$ take this particular form:

$$\frac{a'(\eta)}{a(\eta)}\,\frac{\varphi(x,\eta)}{\phi_0'(\eta)}\ ?$$

Answer:

* The inflaton's inhomogeneities imply locally varying expansion rates.

$\Rightarrow$ some regions are ahead, others lag behind in their expansion.

* Changing the spacetime slicing from a) to b) has to turn pure intrinsic curvature, namely local bloating

$$\frac{\delta a(x,\eta)}{a(\eta)}$$

into pure inflaton fluctuations $\varphi(x,\eta)$.

* Indeed:

$\delta\eta(x)$ is the time "lag" between slicings a) and b)

$$\frac{\delta a}{a} = \frac{1}{a}\frac{\delta a}{\delta \phi}\delta\phi = \frac{1}{a}\frac{\delta a}{\delta\eta}\frac{\delta\eta}{\delta\phi}\delta\phi = \frac{a'}{a}\frac{1}{\phi'}\delta\phi$$

$$= \frac{a'}{a}\frac{1}{\phi'}\varphi \checkmark$$

Ramifications:

- The intrinsic curvature inhomogeneities

very large when ϕ_0' is very small

$$r = -\Psi - \frac{a'}{a}\frac{1}{\phi_0'}\varphi$$

can become strongly enhanced, namely, as it happens, for close to de Sitter inflation:

i.e., for $a(t) \approx e^{Ht}$

i.e., for $H = \frac{\dot a}{a} \approx const$ (recall: $H \sim \sqrt{V(\phi)}$)

i.e., for $\phi \approx const$

i.e., for: $\phi' \approx 0$

Why? Recall that:

$$\frac{\delta a}{a} = \frac{1}{a}\,\frac{\delta a}{\delta\eta}\,\frac{\delta\eta}{\delta\phi}\,\delta\phi = \frac{a'}{a}\,\frac{1}{\phi_0'}\,\ell$$

Thus: Assume $\phi' = \frac{\delta\phi}{\delta\eta} \ll 1$

$$\Rightarrow \quad \frac{\delta\eta}{\delta\phi} \gg 1$$

Intuition:

$\frac{\delta\eta}{\delta\phi} \gg 1$ means that the local time-lag $\delta\eta$ between slicings a.) and b.) is large.

This could mean large $\tau(x,\eta)$ against assumption.

Analogous to: A river in a plain meanders the more widely, the flatter the plain is.

Could it be a problem?

Observations: We know the size of $|r|$ from the CMB. The curvature fluctuations r are of order 10^{-5}. Also, there is evidence that the Hubble radius increased during inflation. Namely, the fluctuations of modes that crossed it late are smaller. So inflation was significantly different from de Sitter.

Is there a preferred slicing of spacetime, say a) or b) in nature?

* Not during inflation, but at its end point!

So it is slicing type a.)

* Why? At each point in space, inflation ends the moment the value of ϕ drops to its minimum. Then, $r(x,\eta)$ is intrinsic curvature.

The expanded action

The action
$$S = \frac{-1}{16\pi G}\int R\sqrt{|g|}\, d^4x$$
$$+\ \frac{1}{2}\int \left((\partial_\nu \phi)(\partial^\nu \phi) - V(\phi)\right)\sqrt{|g|}\, d^4x$$

must be expanded to second order in the inhomogeneities in order to obtain their equations of motion to first order:

$$S = S_S + S_T$$

The scalar part:

$$\boxed{S_S = \frac{1}{2}\int z^2(\eta)\left(\frac{\partial}{\partial x^\mu} r(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu} r(x,\eta)\right)\eta^{\mu\nu}\, d^4x}$$

Here: $\boxed{z(\eta) := \frac{a_0^2(\eta)}{a_0'(\eta)}\,\phi_0'(\eta)} \approx \underbrace{\text{const}\cdot a_0(\eta)}_{\text{because } a_0'(\eta)\approx \text{const}\, a(\eta) \text{ and } \phi_0' \approx \text{const during inflation}}$

Remark:

This action is similar to the scalar action which we considered so far:

$$S_{KG} = \frac{1}{2}\int a^2(\eta)\left(\frac{\partial}{\partial x^\mu}\phi(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x,\eta)\right)\eta^{\mu\nu}\,d^4x$$

The only difference is that $a(\eta)$ is now replaced by the more complicated (but still classical fixed background function) $z(\eta)$.

- The tensor part: Each h_{ij} has exactly our well-known action:

$$\boxed{S_T = \frac{1}{64\pi G}\sum_{i,j=1}^{3}\int a^2(\eta)\frac{\partial}{\partial x^\mu}\left(h^i{}_j(x,\eta)\right)\frac{\partial}{\partial x^\nu}\left(h^i{}_j(x,\eta)\right)\eta^{\mu\nu}\,d^4x \qquad !}$$

Quantization of r and h_{ij}:

- The equations of motion come out to be:

Scalar:

$$r_k''(\eta) + \frac{2z'(\eta)}{z(\eta)}r_k'(\eta) + k^2 r_k(\eta) = 0$$

Tensor:

$$h_{ij_k}''(\eta) + \frac{2a'(\eta)}{a(\eta)}h_{ij_k}'(\eta) + k^2 h_{ij_k}(\eta) = 0$$

- Exercise: verify

- Strategy:

Define auxiliary fields, so that there will be no friction term in the equation of motion.

▯ Recall: Previously in this course, this definition

$$\chi(x,\eta) := a(\eta)\,\phi(x,\eta)$$

achieved an eqn of motion without friction term:

$$\chi_k''(\eta) + \left(k^2 - \frac{a''}{a}\right)\chi_k(\eta) = 0$$

▯ Scalar components:

Since in their action a is replaced by z, we need:

$$u(x,\eta) := -z(\eta)\,r(x,\eta)$$

convenient factor

This yields the eqn. of motion without friction:

$$u_k''(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right)u_k(\eta) = 0$$

▯ The tensor components:

Here, we can define as previously in the course:

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\,a(\eta)\,h_{ij}(x,\eta)$$

convenient factor

to obtain the eqn of motion:

$$p_{ij_k}''(\eta) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right)p_{ij_k}(\eta) = 0$$

Note: * The components of p_{ij} are not all independent, because h_{ij} obeys:

$$h_{ij} = h_{ji} \quad \text{and} \quad \sum_{i=1}^{3} h_{ii} = 0 \quad \text{and in particular:}$$

$$\sum_{i=1}^{3}\frac{\partial}{\partial x^i}h_{ij}(x,\eta) = 0 \quad \text{i.e.} \sum_{i=1}^{3} k_i\,h_{ij}(k,\eta) = 0$$

* But $\vec{k}$ is the vector that points in the direction in which the mode $\vec{k}$ propagates.

$\Rightarrow$ The equation

$$\sum_{i=1}^{3} k_i h_{ij}(k,\eta) = 0$$

(For fixed j, the vectors h_{ij} and k_i are orthogonal) $\longrightarrow$ means that h_{ij} has no component in the propagation direction:

$\Rightarrow$ h_{ij} describes transversal waves (like e.g. tectonic shear waves), not longitudinal waves.

$\rightsquigarrow$ h_{ij} possesses only 2 degrees of freedom:

$v_{k,\lambda}(\eta)$ with $\lambda = 1,2$ or $+$ $\times$

* Polarization decomposition:

$$p_{ij}(k,\eta) := \sum_{\lambda=1,2} v_{k,\lambda}(\eta)\,\varepsilon_{ij}(k,\lambda)$$

Here, $\varepsilon_{ij}(k,\lambda)$ are for each k two arbitrary but fixed matrices, obeying $\sum_{i,j=1}^{3} \varepsilon_{ij}(k,1)\,\varepsilon_{ji}(k,2) = 0$ and:

$$\varepsilon_{ij} = \varepsilon_{ji}\,, \quad \sum_{i=1}^{3} \varepsilon_{ii} = 0\,, \quad \sum_{i=1}^{3} k_i\,\varepsilon_{ij} = 0$$

It is convenient to choose

$$\varepsilon_{ij}(-k,\lambda) = \varepsilon_{ij}^{*}(k,\lambda)$$

because then we have (as usual):

$$v_{k,\lambda}(\eta) = v_{-k,\lambda}^{*}(\eta)$$

$$\Rightarrow \quad v_{k,\lambda}''(\eta) + \left(k^2 - \frac{a''}{a}\right) v_{k,\lambda}(\eta) = 0$$

The goal:

Quantize $\hat{u}_k(\eta)$, $\hat{p}_{ij}(\eta)$ and calculate $\delta r_k(\eta)$ and $\delta h_{ij}(\eta)$

from them at horizon crossing (after which they are constant).

Notice: We cannot simply re-use our de Sitter results b/c Mukhanov variable!

Expectations: * Fluctuations of $\hat{r}$ yield local spacetime expansion (and thus eventually cooling) fluctuations

$\rightarrow$ temperature spectrum in CMB

* Fluctuations of $\hat{h}$ yield grav. waves background. Should appear in polarization spectrum of CMB.

$\rightsquigarrow$ BICEP2 experiment almost found it!