QFT for Cosmology, Achim Kempf, Lecture 22

Note Title

Time line of standard model of cosmology:

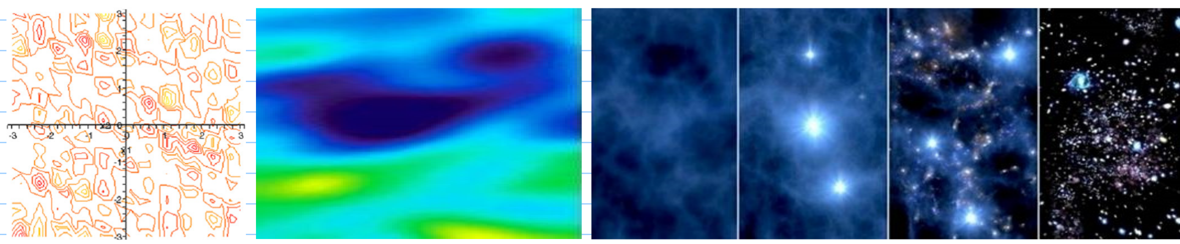
Planck era | Inflation | radiation dominates | matter dominates | "dark energy" Λ dominates

10^{-43}s, 10^{-32}s, 10^{-11}s EW, 10^{0}s nuclei form, 10^{13}s, (10^{5}ys), 10^{19}s today, $\approx \log(t)$

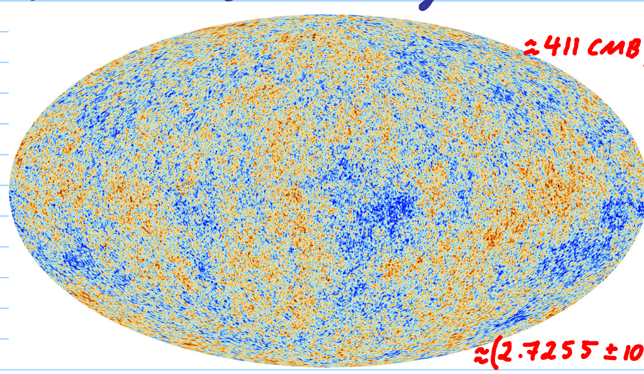
Re-heating

Reverberating expanding and cooling plasma

Atoms form: $p^+ + e^- \rightarrow H$
Universe becomes transparent
Photons travel freely $\rightarrow$ CMB

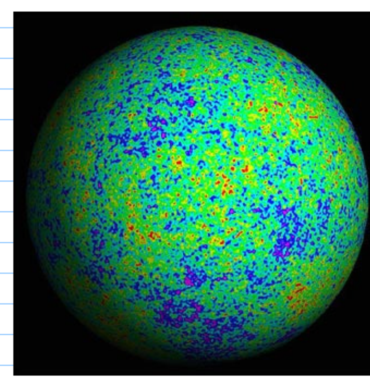


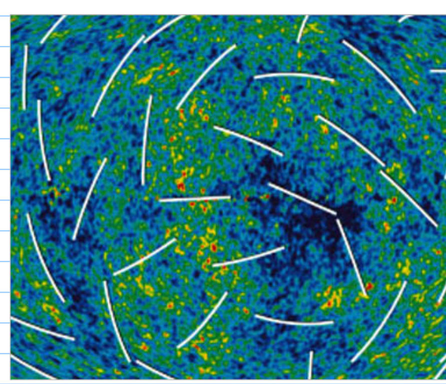
Actual observations of the CMB:



≈ 411 CMB photons/cm^3

$\approx (2.7255 \pm 10^{-5})\,^\circ$K





Zoom-in,
with polarization:
(avg polarization $\approx 10^{-6}$)

Recall:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \quad \text{with} \quad |\varphi(x,\eta)| \ll |\phi_0(\eta)|}$$

$$\boxed{g_{\mu\nu}(x,\eta) = a(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta) \quad \text{with} \quad |\gamma_{\mu\nu}(x,\eta)| \ll 1}$$

$$ds^2 = a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right) + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

$$ds_S^2 = a^2(\eta)\Big[2\Phi(x,\eta)\,d\eta^2 - 2\sum_{i=1}\frac{\partial}{\partial x^i}B(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\left(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}E(x,\eta)\right)dx^i dx^j\Big]$$

$$ds_V^2 = a^2(\eta)\Big[2\sum_{i=1}^{3}V_i(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x^j}W_i(x,\eta) + \frac{\partial}{\partial x^i}W_j(x,\eta)\right)dx^i dx^j\Big]$$

$$ds_T^2 = a^2(\eta)\sum_{i,j=1}^{3}h_{ij}(x,\eta)\,dx^i dx^j$$

The surviving gauge-invariant degrees of freedom are:

- The purely tensorial part of the metric: $h_{ij}(x,\eta)$
- A combination of a scalar part of the metric, $\Psi(x,\eta)$, and $\varphi(x,t)$:

$$r(x,\eta) := -\frac{a_0'}{a_0}\left(\phi_0(\eta)'\right)^{-1}\varphi(x,\eta) - \Psi(x,\eta)$$

They possess these actions:

$$\boxed{S_T = \frac{1}{64\pi G}\sum_{i,j=1}^{3}\int a^2(\eta)\,\frac{\partial}{\partial x^\mu}\left(h^i{}_j(x,\eta)\right)\frac{\partial}{\partial x^\nu}\left(h^i{}_j(x,\eta)\right)\eta^{\mu\nu}\,d^4x}$$

$$\boxed{S_S = \frac{1}{2}\int z^2(\eta)\left(\frac{\partial}{\partial x^\mu}r(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu}r(x,\eta)\right)\eta^{\mu\nu}\,d^4x \quad \text{with} \quad z(\eta) := \frac{a_0^2(\eta)}{a_0'(\eta)}\,\phi_0'(\eta)}$$

To quantize without a friction term, change variable:

$$u(x,\eta) := -z(\eta)\, r(x,\eta)$$

convenient factors

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\, a(\eta)\, h_{ij}(x,\eta)$$

Further, separate of polarization matrices:

$$p_{ij}(k,\eta) := \sum_{\lambda=1,2} v_{k,\lambda}(\eta)\, \varepsilon_{ij}(k,\lambda)$$

⇝ Equations of motion:

$$\hat{v}''_{k,\lambda}(\eta) + \left(k^2 - \frac{a''}{a}\right)\hat{v}_{k,\lambda}(\eta) = 0$$

$$\hat{u}''_k(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right)\hat{u}_k(\eta) = 0$$

Quantum fluctuations

As before, this reduces to solving the eqns of motion for the mode functions, which are complex number-valued, say $\tilde{u}_k(\eta)$, $\tilde{v}_{k,\lambda}(\eta)$:

$$\tilde{u}''_k(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right)\tilde{u}_k(\eta) = 0$$

$$\tilde{v}''_{k,\lambda}(\eta) + \left(k^2 - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0$$

along with the Wronskian conditions.

- **Initial conditions?**

We say we choose the "Bunch Davies vacuum".

At early times:

* The k^2 term dominates

⇒ Can choose Minkowski-like init. conds.

The mode fctns at late times?

At late times:

- The mode k crossed the Hubble horizon:
- The terms $\frac{z''}{z}$ and $\frac{a''}{a}$ dominate.
- The harmonic oscillator is inverted
- Instead of 2 oscillatory basis sols we now expect one growing and one decaying basis solution.
- Soon after horizon crossing the mode function consists of essentially only the growing solution.

Which is the growing solution at late times?

Eqns of motion after horizon crossing:

$$\tilde{u}_k''(\eta) + \left(\cancel{k^2} - \frac{z''(\eta)}{z(\eta)}\right)\tilde{u}_k(\eta) = 0 \text{ , i.e., } \quad \frac{\tilde{u}_k(\eta)''}{\tilde{u}_k(\eta)} = \frac{z(\eta)''}{z(\eta)}$$

$$\tilde{v}_{k,\lambda}''(\eta) + \left(\cancel{k^2} - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0 \text{ , i.e., } \quad \frac{\tilde{v}_{k,\lambda}(\eta)''}{\tilde{v}_{k,\lambda}(\eta)} = \frac{a(\eta)''}{a(\eta)}$$

$\Rightarrow$ Growing solution must behave as:

$$\tilde{u}_k(\eta) \sim z(\eta) \quad \text{at late } \eta$$

$$\tilde{v}_{k,\lambda}(\eta) \sim a(\eta) \quad \text{at late } \eta$$

$\Rightarrow$ The mode fctns $\tilde{r}_k(\eta) = -\frac{\tilde{u}_k(\eta)}{z(\eta)}$ and $\tilde{h}_{ij\,k}(\eta) = 32\pi G \frac{\tilde{v}_{k,\lambda}(\eta)\epsilon_{ij}(k,\lambda)}{a(\eta)}$ become constant at late η, i.e., after the mode k crosses the horizon!

Check: $\tilde{v}_k(\eta) = \frac{1}{z(\eta)}\,\tilde{u}(\eta) \sim \frac{z(\eta)}{z(\eta)}$ for late η

$$\tilde{h}_{ij_k}(\eta) = \frac{1}{a(\eta)}\,\tilde{p}_{ij_k}(\eta) \sim \frac{1}{a(\eta)}\,\tilde{v}_{k,\lambda}(\eta) \sim \frac{a(\eta)}{a(\eta)} \quad \text{for late } \eta$$

$\Rightarrow$ As expected, the magnitude of the mode k's quantum fluctuations

$$\delta r_k = \overbrace{k^{3/2}|\tilde{r}_k|^2}^{z^{-1}k^{3/2}|\tilde{u}_k| \,\shortparallel} \quad \text{and} \quad \delta h_{ij_k} = \overbrace{k^{3/2}|\tilde{h}_{ij_k}|}^{= a^{-1}k^{3/2}|\tilde{v}|}$$

stay constant at the value that they possess when the mode crosses the horizon, even as the mode's proper wavelength then continues to increase rapidly.

* **Goal now:** Calculate the magnitude of the fluctuations at horizon crossing!

Realistic example: "Power law inflation"

- We need an explicit potential $V(\phi)$ in order to be able to find explicit $a(\eta)$, $\phi_0(\eta)$ for which to calculate then the fluctuation spectrum.

- De Sitter is ruled out because:

  * $V(\phi)$, and therefore the temporary "cosmological constant $H \sim \sqrt{V(\phi)}$ must slowly decrease (slow roll).

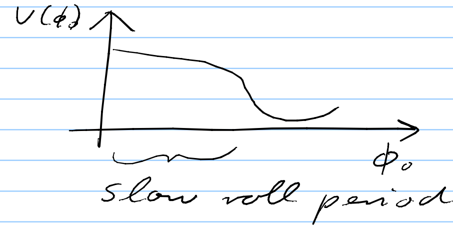
  * In any case, our perturbation assumptions don't allow exact de Sitter, as δr_k would diverge, invalidating the assumption that it is small.

The "slow roll parameters"

Idea:

* We do not know the exact slow roll potential:

E.g.



* For all values of ϕ_0 during the inflationary period we can parametrize the slope of the potential by its derivatives.

* These are the so-called slow roll parameters: (Recall: $H(\phi) \sim \sqrt{V(\phi)}$)

$$\varepsilon(\phi) := \underbrace{\frac{1}{4\pi G}}_{\text{convenience factor}} \left(\frac{H'(\phi)}{H(\phi)}\right)^2 \qquad \left(= \frac{\frac{3}{2}\dot{\phi}^2}{V + \frac{1}{2}\dot{\phi}^2} \right)$$

$$\eta(\phi) := \frac{1}{4\pi G} \frac{H''(\phi)}{H(\phi)} \qquad \left(= \varepsilon - \frac{\varepsilon'}{\sqrt{16\pi G \varepsilon}} \right)$$

$$\xi(\phi) := \frac{1}{4\pi G} \sqrt{\frac{H'(\phi) H'''(\phi)}{H^2(\phi)}}$$

etc...

The simplest solvable case:

* The simplest case is that of

$$\varepsilon(\phi) = c \quad \text{where } c \text{ is a constant.}$$

* In this case:

$$c = \varepsilon(\phi) := \frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2$$

Thus,

$$H(\phi) \sim e^{\sqrt{4\pi G c}\,\phi}$$

and the potential is of the form:

$$V(\phi) = e^{s\phi}$$

* Exercise: What is the value of s ?

* Then, one also finds:

$$c = \varepsilon = \eta = \xi = \dots$$

* The expansion rate is polynomial:

Exercise:

Show that:

$$a(t) = a_0\, t^{1/c}$$ (t is proper time)

Exercise:

Show that, in terms of the conformal time η:

$$a(\eta) = \frac{-1}{\eta H}\,\frac{1}{1-\varepsilon}$$

Note: Still η is always negative and $t \to \infty$ means $\eta \to 0$.

The mode equations:

Scalar: We can now calculate $z(\eta) = \frac{a^2(\eta)}{a'(\eta)} \phi_0'(\eta)$ and therefore also the mode equation's term z''/z explicitly, to obtain

A Bessel differential equation

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{(\nu^2 - 1/4)}{\eta^2}\right) \tilde{u}_k(\eta) = 0$$

where: $\nu := \frac{3}{2} + \frac{c}{1-c}$

* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu + 1/2)\frac{\pi}{2}} (-\eta)^{1/2} H_\nu^{(1)}(-k\eta)$$

Hankel fctn of 1st kind of order ν.

* Behavior after horizon crossing:

$$\tilde{u}_k(\eta) \to e^{i(\nu - \frac{1}{2})\frac{\pi}{2}} 2^{\nu - 3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{-\nu + 1/2}$$

* Thus, the magnitude of intrinsic curvature fluctuations after horizon crossing becomes:

$$\boxed{\delta r_k\,(\eta > \eta_{hor}(k)) = G\, 2^{\nu - \frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \left(\nu - \tfrac{1}{2}\right)^{1/2 - \nu} \frac{H^2}{|H'|}\Bigg|_{\text{at } k = aH}}$$

horizon crossing

Exercise: verify

Notice: Measurement of δr_k in CMB can only tell us $\frac{H^2}{H'}$ (at horizon crossing) but not H or H' individually!

Intuition?

Earlier, for a K.G. field ϕ in a fixed background FRW universe, we found:

$$\delta\phi_k \sim H$$

Now, for the intrinsic curvature (the Mukhanov variable), we found:

$$\delta r_k \sim H^2/|H'|.$$

Recall: r_k is the slicing-independent combination of the scalar part of

$\delta g_{\mu\nu}$ and ϕ.

The slower the roll ($|H'|$ small) the wider away from another fluctuate gauge equivalent and inequivalent slicings:

Analogous to: A river in a plain meanders the more widely, the flatter the plain is.

Recall the timeline:

Planck era ? | Inflation | radiation dominates | matter dominates | A Λ, inducing a "dark energy" dominates

10^{-43}s — 10^{-32}s — 10^{-11}s — 10^{0}s nuclei form — 10^{13}s — (10^{5}ys) — 10^{19}s today — $\approx \log(t)$

Re-heating

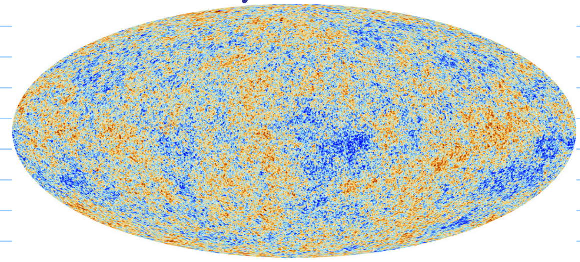
Reverberating expanding and cooling plasma

known plasma physics there's code for that step.

Atoms form: $p^+ + e^- \to H$
Universe becomes transparent
Photons travel freely → CMB

we calculated: δr_K at horizon crossing, then unchanged till end of inflation: re-heating.

Observations by WMAP, Planck et al:



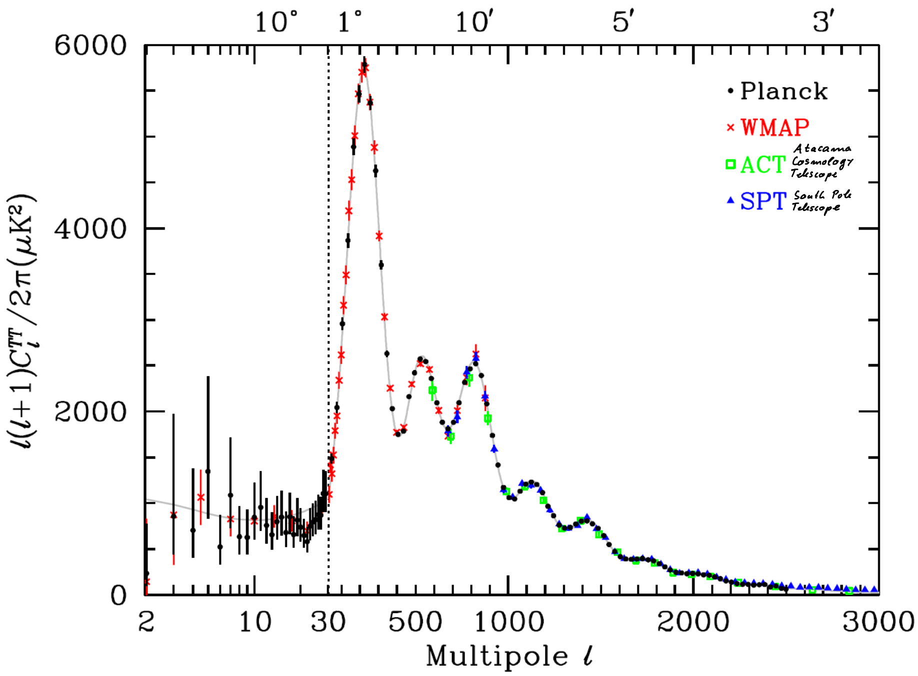
- δr_k is predicted to have seeded oscillations in the hot plasma after re-heating. The plasma decohered the quantum fluctuations of the intrinsic curvature r.

- Standard plasma physics allows one to calculate the propagation and dispersion for the $\approx 10^5$ ys until hydrogen formed.

- The temperature fluctuation spectrum in the CMB is from gravitational blue and redshifts due to these curvature fluctuations.

- Theory matches experiment closely, while fixing cosmological parameters, including indications that $\varepsilon \neq 0$, namely that $\delta r_k \neq$ const.

Best fit today:



$K = 0$

$\Lambda \approx 0.7\, \rho_{\text{critical}}$

$\rho_{\text{matter}} \approx 0.3\, \rho_{\text{critical}}$

$\rho_{\text{dark matter}} \approx 0.9\, \rho_{\text{matter}}$

$\rho_{\text{visible matter}} \approx 0.1\, \rho_{\text{matter}}$

$\rho_{\text{CMB}} \approx 5 \cdot 10^{-5}\, \rho_{\text{critical}}$

$v_{\text{peculiar of earth}} \approx 370$ km/s

II Tensor modes:

$$\tilde{v}''_{k,\lambda} + \left(k^2 - \frac{a''}{a}\right)\tilde{v}_{k,\lambda} = 0$$

we obtain for the term a''/a:

$$\frac{a''}{a} = 2a^2H^2\left(1 - \varepsilon/2\right)$$

which comes out to be (verify):

$$\frac{a''}{a} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)$$

recall: $\nu = \frac{3}{2} - \frac{c}{1-c}$

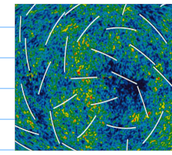
$\Rightarrow$ The mode eqn is again solved by the Hankel function.

$\Rightarrow$ The tensor fluctuation spectrum:

horizon crossing.

$$\boxed{\delta h_{ij_k} = \frac{2}{\sqrt{\pi}}\, 2^{\nu - \frac{1}{2}}\, \frac{\Gamma(\nu)}{\Gamma(3/2)}\, (\nu - 1/2)^{1/2 - \nu} \sqrt{G}\; H \Big|_{k = aH}}$$

δh_{ij} should have left curl ("B") polarization in the CMB

Experiments show polarization in the CMB: 

- But most is gradient ("E") polarization that originated in δr_R or in foreground.
- So far, h_{ij}-originated B-polarization cannot be distinguished from foreground.
- Observation of h_{ij} polarization:
  - * Would show quantized gravitational waves!
  - * Would determine the scale of H, and therefore of H'!
  - * This would tell the slope of the spectra
    - $\Rightarrow$ Nontrivial consistency conditions to check inflation.