Note Title

QFT for Cosmology, Achim Kempf, Lecture 25

Recall strategy:

we continue here

SR, 1^{st}Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, 1^{st}Q. Lagrangian formalism

step 2: allow curvature

GR, 1^{st}Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1^{st}Q Lagrangian formalism

step 4

GR, 2^{nd}Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, 2^{nd} Q Lagrangian formalism (Path integral of QFT)

2^{nd} quantization using Feynman's path integral

- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system.

- Then, for each field $\phi(\vec{x},t)$ we can calculate its action $S[\phi,g]$, e.g., for the Klein Gordon fields:

$$S_{KG}[\phi,g] = \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \lambda\phi^4\right)\sqrt{|g|}\, d^4x$$

- Following Feynman, we obtain a "probability amplitude"

$$\text{prob. ampl.}[\phi] := \frac{1}{c}\, e^{\frac{i}{\hbar}S_{KG}[\phi,g]}$$

we'll drop writing these

for any particular field evolution $\phi(x,t)$ to occur.

- Note: $w[\phi] = e^{iS[\phi]}$ is an un-normalized prob. amplitude

Therefore, c is the normalization constant.

- Recall:

  - Assume given unnormalized probabilities $w(A_i)$ for some value A_i to be found.

  - Then, the expectation value $\bar{A}$ is given by:

$$\bar{A} = \frac{1}{c} \sum_i A_i\, w(A_i) \quad \text{where } c = \sum_i w(A_i)$$

  - Thus:

$$\bar{A} = \frac{\sum_i A_i\, w(A_i)}{\sum_i w(A_i)}$$

$\Longrightarrow$ In QFT, now that we can calculate probability amplitudes, we should be able to calculate all expectation values!

Example 1: Average field amplitude

Note:
The path integral is ill-defined analytically. But, algebraically it yields a consistent algorithm for calculating expectation values. One expects that there are UV and IR cutoffs in nature which render the path integrals of QFT also analytically well-defined.

$$\bar{\phi}(x) = \frac{\int \overbrace{\phi(x)}^{\text{odd in } \phi} e^{iS[\phi]}\, D[\phi]}{\int e^{iS[\phi]}\, D[\phi]}$$

"Path Integral", i.e. integral over all variables $\phi(x)\ \forall x$. for $\phi(x) \in (-\infty, \infty)$

$\leftarrow$ Normalization

Assume e.g:

$$\underbrace{S[\phi]}_{\text{is then even in } \phi} = \int \frac{1}{2} \phi(x)(\Box - m^2)\phi(x) + \frac{\lambda}{4!} \phi^4(x)\, d^4x$$

$\Longrightarrow$ $\bar{\phi}(x) = 0$ We obtained this very quickly!

Compare: How did we calculate $\bar{\phi}(x)$ previously?

$$\bar{\phi}(x) = \langle 0 | \hat{\phi}(x) | 0 \rangle$$

Notice that we can say more when using the path integral approach, and more quickly.

Hard to evaluate except if $\lambda = 0$. Then:

$$\hat{\phi}(x) = \int e^{ikx}\left(u_k(t)\, a_k + u_k^*(t)\, a_k^+\right) d^3k$$

Thus: $\bar{\phi}(x) = \langle 0 | \ast a + \ast a^+ | 0 \rangle = 0$

Example 2: 2-point correlation function

$$G^{(2)}(x,x') := \frac{\int \phi(x)\,\phi(x')\, e^{iS[\phi]}\, D[\phi]}{\int e^{iS[\phi]}\, D[\phi]}$$

* **Meaning of $G^{(2)}(x,x')$?**

It shows how much the field amplitudes at events x and x' are correlated over some spatial and temporal distance.

* **How to calculate $G^{(2)}(x,x')$ with old methods?**

$$G^{(2)}(x,x') = \langle 0 | \hat{\phi}(x)\, \hat{\phi}(x') | 0 \rangle \qquad \text{(A)}$$

Here, we assume for now that: $x_0 \geq x_0'$.

* In space times where we have an explicit mode decomposition, e.g., FRW,

$$\hat{\phi}(x) = \int e^{ikx}\left(u_k(t)\, a_k + u_k^*(t)\, a_k^+\right) d^3k \qquad \text{(B)}$$

Recall: This means that the result depends on the identification of the vacuum state

we obtain $G^{(2)}(x,x')$ in terms of $u_k(t)$.

* Why $x_0 \geq x_0'$?

We have $[\hat\phi(x), \hat\phi(x')] = 0$ if $x_0 = x_0'$

but, in general, they don't commute.

Opposite choice could be made but -1 would have to be absorbed somewhere.

We choose: Earlier is always right, later is left.

* To automatize the bookkeeping, define T:

$$G^{(2)}(x,x') := \langle 0 | T\, \hat\phi(x)\,\hat\phi(x') | 0 \rangle$$

↑ "Time ordering operator"

$$= \langle 0 | \, \Theta(x_0 - x_0')\,\hat\phi(x)\,\hat\phi(x')$$
$$+ \Theta(x_0' - x_0)\,\hat\phi(x')\,\hat\phi(x) \, | 0 \rangle$$

↑ Heaviside step function

Remarks:

* From Eqs. A, B follows $G^{(2)}(x,x')$.

* $G^{(2)}(x,x')$ is nonzero almost everywhere!

* If x, x' are in each other's lightcone then correlation of $\phi(x)$, $\phi(x')$ can be due to the propagation of the amplitude from one event, say x, to the other, x'.

* But $G^{(2)}(x,x') \neq 0$ even if x, x' are spacelike separated, e.g., if $x_0 = x_0'$!

This cannot be due to communication, i.e., propagation. Indeed: $|0\rangle$ is entangled!

We saw how to calculate $G^{(2)}$ using old methods.

How to calculate $G^{(2)}$ using the path integral?

$$G^{(2)}(x,x') := \frac{\int \phi(x)\,\phi(x')\, e^{iS[\phi]}\, D[\phi]}{\int e^{iS[\phi]}\, D[\phi]}$$

Use Dyson Schwinger method:

Consider integral of a total derivative:

$$\int \frac{\delta}{\delta\phi(x)} \left(\phi(x')\, e^{iS[\phi]}\right) D[\phi]$$

$$= \text{boundary term}$$

$$= 0$$

Thus:

$$0 = \int \frac{\delta}{\delta\phi(x)} \left(\phi(x')\, e^{iS[\phi]}\right) D[\phi]$$

$$= \int \left(\delta^4(x-x') + \phi(x')\, i \frac{\delta S}{\delta\phi(x)}\right) e^{iS[\phi]}\, D[\phi]$$

(assume now $\lambda = 0$) $\Rightarrow$

$$= \int \Big(\delta^4(x-x') + i\phi(x') \overbrace{\sqrt{g}\,(\Box_x - m^2)\,\phi(x)}^{=}\Big) e^{iS[\phi]}\, D[\phi]$$

Nontrivial on curved space: Includes $\sqrt{g}$.

$$\Rightarrow \quad 0 = \delta^4(x-x') \int e^{iS[\phi]}\, D[\phi] + i\sqrt{g}\,(\Box_x - m^2) \int \phi(x)\,\phi(x')\, e^{iS[\phi]}\, D[\phi]$$

$\times \dfrac{1}{\int e^{iS[\phi]} D[\phi]}$

$$\Rightarrow \quad \boxed{\sqrt{g}\,(\Box_x - m^2)\, G^{(2)}(x,x') = i\,\delta^4(x-x')}$$

Thus, $G^{(2)}$ is called a Green's function for the Klein Gordon equation.

Exercise: Show that indeed $\sqrt{g}\,(\Box_x - m^2)\langle 0|T\,\hat\phi(x)\,\hat\phi(x')|0\rangle = \delta^4(x-x')$

Hint: $(\Box_x - m^2)\hat\phi(x) = 0$ of course, but ∂_t in $\Box$ acts nontrivially on $\theta(t-t')$ in T.

Note:

- For any given metric g this becomes an explicit partial differential equation.
- If we can solve it, we know all about the propagation and also all about the correlations caused by the entanglement of the vacuum state.
- But is the solution unique?

  Does the operator $(\Box_x - m^2)$ have a unique right inverse?
- No! Because $\exists$ solutions, to $(\Box_x - m^2)\,G^{(2)}_{hom}(x,x') = 0$!

- Example Minkowski space:

  Fourier transform spatial coordinates, to obtain:

$$\left(\partial_t^2 + \vec{k}^2 + m^2\right)\, G^{(2)}(t,t',k) = i\,\delta(t-t')$$

  Thus, $G^{(2)}(t,t',k)$ is unique only up to the choice of a solution to

$$(\partial_t^2 + \vec{k}^2 + m^2)\, G^{(2)}_{hom}(t,t',k) = 0$$

  These are the mode solutions:

$$G^{(2)}_{hom}(t,t',k) = u_k(t-t')$$

  $\Rightarrow$ The ambiguity in $G^{(2)}$ corresponds to the ambiguity in fixing the vacuum state.

The central quantities in elementary particle physics:

- Of central significance in quantum field theory are the fields' n-point correlation functions:

$$G(x_1, x_2, \dots, x_n) := N \int_{\text{all } \phi} \phi(x_1)\,\phi(x_2)\dots\phi(x_n)\, e^{iS[\phi]}\, D[\phi]$$

with, as before: $N^{-1} = \int e^{iS[\phi]} D[\phi]$

- Here: * Each x_i is a point in spacetime, i.e., an event.

  * The integral is formally the sum over all (differentiable, square integrable etc) fields ϕ.

Proposition: $G^{(n)}(x_1, \dots, x_n) = \langle 0 | T\, \hat{\phi}(x_1) \cdot \dots \cdot \hat{\phi}(x_n) | 0 \rangle$

Proof strategy: Show that LHS and RHS obey same differential equation.

- Case $n=2$:

  ✓ $(\Box_x + m^2)\, G^{(2)}(x,x') = i\,\delta^4(x-x')$ (we showed)

  ✓ $(\Box_x + m^2) \langle 0 | T \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle = i\,\delta^4(x-x')$ (given as exercise)

- For general n: "Schwinger-Dyson equations"

  From path integral, easy to derive, generalizing above ansatz which worked for $n=2$:

$$0 = \int \frac{\delta}{\delta\phi(x)} \left(\phi(x_2) \cdot \dots \cdot \phi(x_n)\, e^{iS[\phi]} \right) D[\phi]$$

... then work out the derivatives.

- It is much more work in the operator formalism to derive that the $\langle 0|T\,\hat\phi(x_1)\cdots\hat\phi(x_n)|0\rangle$ obey the same equations. But it can be done.

- Why cumbersome in operator framework?

  Recall how ∂_t^2 in $\Box$ affects θ of T in $\langle 0|T\hat\phi\hat\phi|0\rangle$.

Note:
In the "imaginary time" formalism, the KG eqn is elliptic, thus has no homogeneous solutions $\Rightarrow G^{(2)}$ unique in P.I. approach. Then, analytic continuation to real time yields a unique choice – and generally the right choice. Problem: Imaginary time formalism not generally available on curved spacetimes.

- Thus, QFT in operator and PI formalism yield same predictions for the correlation functions (and everything else), provided the Dyson Schwinger equations are solved with the same initial conditions.
(i.e., same vacuum)

- Notice: PI approach does not fix the initial conditions
(and thus the vacuum state)

What PI approach is best for: Perturbation theory for interactions

* We introduce an auxiliary field, J, called a "source field":

$$G(x_1, x_2, \ldots, x_n) := N\int_{\text{all }\phi} \phi(x_1)\,\phi(x_2)\ldots\phi(x_n)\, e^{iS[\phi]}\, D[\phi]$$

Source field

$$= (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)}\, N\int_{\text{all }\phi} e^{iS[\phi] + i\int J(x)\phi(x)\,d^4x}\, D[\phi]\Bigg|_{J=0}$$

* Let us define an action, $\tilde S[\phi, J]$, that includes the sources:

$$\tilde S[\phi, J] := \int_{\mathbb{R}^4} \overbrace{\frac{1}{2}\phi(x)\left(\frac{\partial^2}{\partial x^{0\,2}} - \Delta + m^2\right)\phi(x) + \frac{\lambda}{8}\phi(x)^4}^{\text{Recall: } = S'[\phi]} + J(x)\phi(x)\, d^4x$$

* **Definition:**

The "partition function", $Z[J]$, is defined as:

$$Z[J] = N \int_{\text{all } \phi} e^{i\tilde{S}[\phi, J]} \, D[\phi]$$

* We can now write the correlation functions in this form:

$$G(x_1, \dots, x_n) = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \, Z[J] \Big|_{J=0}$$

How can we now calculate the n-point functions?

* To this end, it obviously suffices to calculate $Z[J]$, since $Z[J]$ is "generating functional" for the $G(x_1, \dots, x_n)$.

* Explicitly, e.g., for $\lambda \phi^4$-theory:

$$Z[J] = N \int_{\text{all } \phi} e^{i\tilde{S}[\phi, J]} \, D[\phi]$$

$$= N \int_{\text{all}\,\phi} e^{i\int_{\mathbb{R}^4} \frac{1}{2}\phi(x)\left(\frac{\partial^2}{\partial x^{0\,2}} - \Delta + m^2\right)\phi(x) + \frac{\lambda}{8}\phi(x)^4 + J(x)\phi(x) \, d^4x} \, D[\phi]$$

Note: this step is also analytically ill defined. $\longrightarrow$

$$= e^{i\int \frac{\lambda}{8}\left(\frac{\delta}{\delta J(\tilde{x})}\right)^4 d^4\tilde{x}} \; N \int_{\text{all}\,\phi} e^{i\int_{\mathbb{R}^4} \frac{1}{2}\phi(x)\left(\frac{\partial^2}{\partial x^{0\,2}} - \Delta + m^2\right)\phi(x) + J(x)\phi(x) \, d^4x} \, D[\phi]$$

* Introduce a shorthand matrix & vector notation:

$$\int_{\mathbb{R}^4} \frac{1}{2}\phi(x)\left(\frac{\partial^2}{\partial x^{0\,2}} - \Delta + m^2\right)\phi(x)\, d^4x + \int_{\mathbb{R}^4} J(x)\phi(x)\, d^4x$$

With suitable choice of a countable basis in the function space of the fields:

$$= \sum_{i,j} \frac{1}{2} f_i M_{ij} f_j + \sum_k j_k f_k$$

$$= \frac{1}{2} f M f + j f$$

* Thus:

$$Z[J] = e^{i\int \frac{\lambda}{8}\left(\frac{\delta}{\delta J(\tilde{x})}\right)^4 d^4\tilde{x}}\, N \int_{\text{all } \phi} e^{i\int_{\mathbb{R}^4} \frac{1}{2}\phi(x)\left(\frac{\partial^2}{\partial x^{0\,2}} - \Delta + m^2\right)\phi(x) + J(x)\phi(x)\, d^4x}\, D[\phi]$$

$$= e^{i\frac{\lambda}{8}\left(\frac{\delta}{\delta j}\right)^4}\, N \int_{\text{all } f} e^{i\left(\frac{1}{2} f M f + j f\right)}\, D[f]$$

* We observe: (completion of squares)

$$\frac{1}{2} f M f + j f = \frac{1}{2}\left(f + M^{-1} j\right)^t M \left(f + M^{-1} j\right) - \frac{1}{2} j M^{-1} j$$

* Thus:

$$Z[J] = e^{i\frac{\lambda}{8}\left(\frac{\delta}{\delta j}\right)^4}\, N \int_{\text{all } f} e^{i\frac{1}{2}\left(f + M^{-1} j\right)^t M \left(f + M^{-1} j\right) - \frac{i}{2} j M^{-1} j}\, D[f]$$

change integration variable: $\tilde{f} := f + j M^{-1}$ $\Rightarrow$

$$= e^{i\frac{\lambda}{8}\left(\frac{\delta}{\delta j}\right)^4} e^{-\frac{i}{2} j M^{-1} j}\, \overbrace{N \int e^{i\frac{1}{2}\tilde{f} M \tilde{f}}\, D[\tilde{f}]}^{=:\,\tilde{N}}$$

$$= \tilde{N}\, e^{i\frac{\lambda}{8}\left(\frac{\delta}{\delta j}\right)^4} e^{-\frac{i}{2} j M^{-1} j}$$

Normalization constant

* Back in original notation:

$$Z[J] = \tilde{N}\, e^{\frac{i\lambda}{8}\int_{\mathbb{R}^4}\left(\frac{\delta}{\delta J(\tilde{x})}\right)^4 d^4\tilde{x}}\; e^{-\frac{i}{2}\int_{\mathbb{R}^4} J(x)K(x,x')J(x')\,d^4x\,d^4x'}$$

* Recall that:

$$M = \partial_t^2 - \Delta + m^2$$

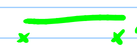
* Thus:

$K = M^{-1}$ is a Green's function:

* I.e., K obeys:

$$(\partial_t^2 - \Delta + m^2)\,K(x,x') = \delta^4(x-x')$$

* Here, $K(x,x')$ is called the **propagator**.

* $K(x,x')$ will be represented by an edge .

Where graphs come in:

* The generating function

$K(x,x')$ is known explicitly (e.g., Hankel fctns)

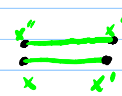
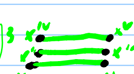
$$Z[J] = \tilde{N}\, e^{\frac{i\lambda}{8}\int_{\mathbb{R}^4}\left(\frac{\delta}{\delta J(\tilde{x})}\right)^4 d^4\tilde{x}}\; e^{-\frac{i}{2}\int_{\mathbb{R}^4} J(x)K(x,x')J(x')\,d^4x\,d^4x'}$$

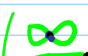
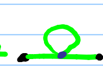
$$\left(\frac{\delta J(x)}{\delta J(x')} = \delta^4(x-x')\right)$$

$$= \tilde{N}\left(1 + \frac{i\lambda}{8}\int_{\mathbb{R}^4}\left(\frac{\delta}{\delta J(\tilde{x})}\right)^4 d^4\tilde{x} + \dots\right)\left(1 - \frac{i}{2}\int_{\mathbb{R}^4} J(x)K(x,x')J(x')\,d^4x\,d^4x' + \dots\right)$$

may now be viewed as a sum of graphs: (all x, x' etc integrated)

$$\int d^4\tilde{x}\; K(\tilde{x},\tilde{x})^2$$

$= \tilde{N}\Big[\, 1 - \frac{i}{2}$  $+ \frac{1}{2!}\left(\frac{-i}{2}\right)^2$  $+ \frac{1}{3!}\left(\frac{-i}{2}\right)^3$  $+ \dots$

$+ \frac{i\lambda}{8}\Big(\emptyset + \frac{c_1}{2!}\left(\frac{i}{2}\right)^2$  $+ \frac{c_2}{3!}\left(\frac{-i}{2}\right)^3\Big($  $+$  $+ \dots\Big) + \dots \Big]$

* We have:

  - One kind of edge:

    $\underset{x}{}\!\!\text{———}\!\!\underset{x'}{} = K(x,x')$ "propagator"

  - Two kinds of vertices:

    $\underset{x}{\bullet} = \frac{i}{2} J(x)$ A free end: an ingoing or outgoing particle

    $\underset{x}{\bullet} = \frac{i\lambda}{8} \delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\delta^4(x-x_4)$

    This vertex describes the collission of particles

* Note: $Z[J]$ contains connected and disconnected graphs.

* Definition: Let $iW[J]$ be the sum of only all connected graphs.

* Proposition: We have:

$$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \left(iW[J]\right)^N = e^{iW[J]}$$

This is clear because:

  - disconnected graphs are products of connected graphs
  - the factor $1/N!$ avoids overcounting because in $Z[J]$ the order of the connected subgraphs does not matter (since the vertices can be re-labeled).

* Thus: $W[J] = -i \ln Z[J]$ is the generating functional for the connected graphs.

Outlook:

Why work with $W[J]$, instead of $Z[J] = e^{iW[J]}$?

Recall: $Z[J] = N \int_{all\,\phi} e^{i\tilde{S}[\phi,J]} D[\phi]$ i.e.: $Z[0] = 1$ means

N^{-1} = sum over all graphs without end vertices, i.e., it is the sum over all disconnected graphs.

Thus:

$$(i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0}$$

inherits N and the disconnected graphs but in

$$(i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J]\Big|_{J=0}$$

they cancel!

Exercise: Show this, using that $(\ln f)' = \frac{f'}{f}$