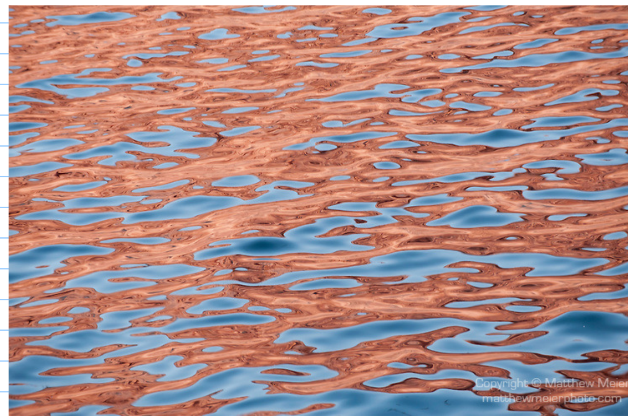
Note Title

QFT for Cosmology, Achim Kempf, Lecture 2

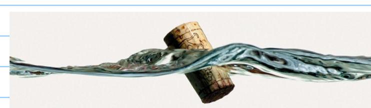
A taste of quantum fields

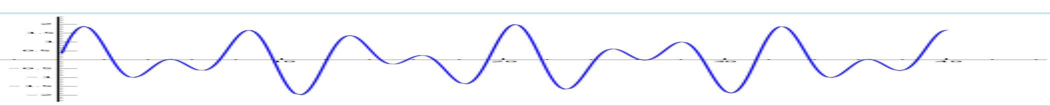
Intuition:

* Consider water waves:

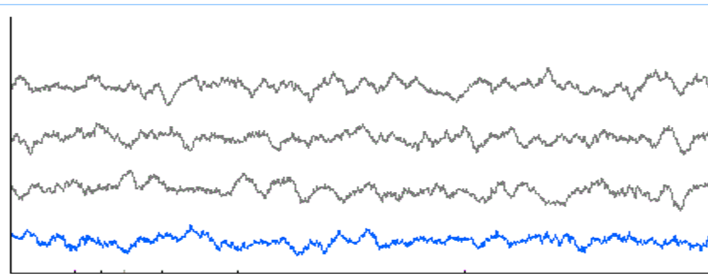


* Probe them locally with cork:



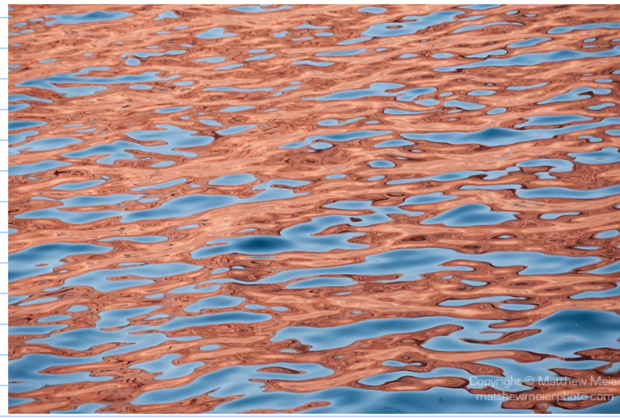


* Multiple cork's oscillations are correlated



not harmonic for water, not quite in QFT either.

$\rightsquigarrow$ System of coupled (harmonic) oscillators !



Plan:

1. Recall harmonic oscillators
2. Relativistic fields
3. 2nd quantization
4. The harmonic oscillators of fields & their vacuum fluctuations

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1. Harmonic oscillators

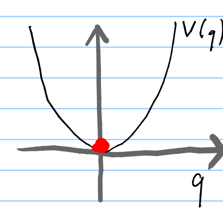
Classical:

- Hamiltonian: $H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$

- Eqns of motion: $\dot{p} = -\omega^2 q, \quad \dot{q} = p$

- Lowest energy solution: (later relevant for "vacuum")

  $q(t) = 0, \; p(t) = 0$
  i.e., $H(t) = 0$ for all t:



- "Nothing moves, with certainty"

Quantum:

As always when quantizing:

- H and Eqns of motion unchanged.

- But, the canonically conjugate pairs of variables (here, q and p) no longer commute:

- Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}\hat{q}^2$

- Eqns of motion: $\dot{\hat{p}} = -\omega^2 \hat{q}, \quad \dot{\hat{q}} = \hat{p}$

- And now:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

- $\Rightarrow$ $\hat{q}(t)$, $\hat{p}(t)$, $\hat{H}$ etc are operator-valued.

- Lowest energy solution now?

  The lowest energy state, $|\psi_0\rangle$, obeys:

$$\hat{H}|\psi_0\rangle = E_0 |\psi_0\rangle$$

  with $E_0 = \frac{1}{2}\hbar\omega$

- We notice:

  Lowest energy is elevated! Why?

  (Later for quantum fields $\Rightarrow$ nonzero vacuum energy)

Lowest energy state $|\psi_0\rangle$?

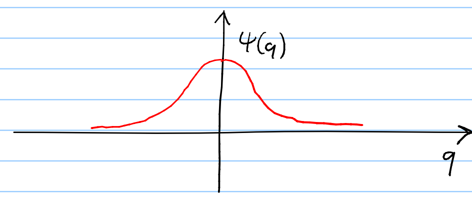
Consider eigenbasis $|q\rangle$ of $\hat{q}$:

$$\hat{q}|q\rangle = q|q\rangle \quad \text{for } q \in \mathbb{R}$$

$$\langle q|q'\rangle = \delta(q-q')$$

Then, recall:

$$\psi_0(q) = \langle q|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar}q^2}$$



Is oscillator at resting position $q=0$?

In lowest energy state, $|\psi_0\rangle$, we have:

$$\bar{q} = \langle\psi_0|\hat{q}|\psi_0\rangle = \int_{-\infty}^{+\infty} \psi_0^*(q)\, q\, \psi_0(q)\, dq = 0$$

i.e. the position expectation vanishes, as in classical mechanics.

But, there are quantum fluctuations!

$$\Delta q = \langle\psi_0|(\hat{q}-\bar{q})^2|\psi_0\rangle^{1/2} = \sqrt{\frac{\hbar}{2m}}$$

i.e., actual measurements yield values spread around $q=0$.

$\Rightarrow$ plausible why energy is elevated

Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields
3. 2nd quantization
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

2. Relativistic fields

- How to make the Schrödinger equation, say

choose simple case without a potential

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\,\psi(x,t) \qquad (S)$$

Laplacian: $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$

relativistically covariant?

- **Klein & Gordon:**

Recall: $p_j = -i\hbar \frac{\partial}{\partial x_j}$ and $E = i\hbar \frac{\partial}{\partial t}$ (because $= \hat{H} =$ energy), i.e., the Schrödinger equation can be written in this form:

$$E\psi = \frac{\vec{p}^{\,2}}{2m}\psi, \text{ i.e.:} \qquad \boxed{E = \frac{\vec{p}^{\,2}}{2m}}$$

$\vec{p}^{\,2} = \sum_{i=1}^{3} p_i^2$ — i.e. $E = \frac{1}{2}m\dot{x}^2$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2 \qquad (\text{Namely: } p_\mu p^\mu = m^2c^4)$$

i.e.:

$$\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\Delta\right)\psi = m^2c^2\psi$$

This "Klein Gordon equation" is usually written as:

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\psi = 0}$$

(units chosen so that $c=1$, $\hbar=1$)

Or, also $(\Box + m^2)\psi = 0$ with d'Alembertian $\Box = \partial_t^2 - \Delta$

Nonrelativistic limit ok?

Must show that KG eqn reduces to Schrödinger eqn for small momenta:

Assume K.G. Eqn., i.e.,: $\frac{E^2}{c^2} = m^2c^2 + \vec{p}^{\,2}$

$$\Rightarrow \quad E = \pm\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$$

Choose positive energy solution:

$$E = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$$

Taylor expansion for small $\vec{p}^{\,2}$: (or large c)

$$E = mc^2 + \frac{1}{2}\frac{c^2}{\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}}\Bigg|_{\vec{p}^{\,2}=0} \vec{p}^{\,2} + \mathcal{O}((\vec{p}^{\,2})^2)$$

$$\Rightarrow \quad E = mc^2 + \frac{\vec{p}^{\,2}}{2m} + \mathcal{O}((\vec{p}^{\,2})^2)$$

$\Longrightarrow$ For small momenta the K.G. eqn becomes the Schrödinger eqn:

$$E\psi = \left(\frac{\vec{p}^{\,2}}{2m} + mc^2\right)\psi$$

i.e.: $$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\Delta + mc^2\right)\psi$$

Note: We obtain an extra term:

$$\hat{H} = \frac{\hat{p}^2}{2m} + mc^2$$

In QM irrelevant: (use Heisenberg picture)

$$i\hbar\frac{d}{dt}\hat{f} = [\hat{f}, \hat{H} + \text{const}\,1] = [\hat{f}, \hat{H}]$$

Remarks:

1a) The negative energy solutions spoil the interpretation of the $\psi(x,t)$ as a probability amplitude density!

Namely: Require the negative energy solutions to propagate backwards in time: anti-particles! They look like travelling forward in time with opposite properties.

1b) This problem is deep and led to quantum field theory, where this is solved in terms of anti-particles.

2a) There are <u>many</u> ways to generalize the Schrödinger equation to obtain a relativistically covariant equation.

2b) E. Wigner (1940s): Complete classification of relativistically covariant wave equations:

Note: The complete classification allows arbitrarily high spins and distinguishes massive from massless cases. All covariant wave eqns for same spin and mass lead to equivalent QFTs. See, e.g., textbook on QFT by S. Weinberg.

Spin	Standard wave eqn	Examples
0	Klein Gordon eqn.	Higgs, Inflaton, $\pi^0, \pi^\pm$
1/2	Dirac eqn.	e^-, quarks, p^+, n
1	Maxwell YM eqns.	Photons, gluons

Higher spins?

- not observed in truly elementary particles.
- appear to lead to incurable "divergencies" in QFT.

Note:

- "Graviton" should be a spin 2 particle.

Plan:

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3. 2nd quantization

- We will 2nd quantize only the Klein Gordon equation because:
  - is easiest
  - is only case of cosmological significance that we know of (so far).

- Terminology: We switch from ψ to ϕ and call it a "Field".

- Definition:

we will do the general definition later

The canonically conjugate field $\pi(x,t)$ to $\phi(x,t)$ is defined as: $\pi(x,t) = \dot{\phi}(x,t)$ (analogous to $p_i = \dot{q}_i$)

- Klein Gordon equation can now be written in the form:

$$\dot{\pi}(x,t) - \Delta\phi(x,t) + m^2\phi(x,t) = 0$$

Notice:

The K.G. equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi = 0 \qquad (\hbar = 1 = c)$$

does not couple $\mathrm{Re}(\phi)$ to $\mathrm{Im}(\phi)$:
each separately fulfills the K.G. eqn.

$\Rightarrow$ It suffices to study real-valued ϕ.

Making ϕ complex is then straightforward.

- Quantization conditions:

$$\left[\hat{\phi}(x,t), \hat{\pi}(x',t)\right] = i\hbar\, \delta^3(x-x')$$

$$\left[\hat{\phi}(x,t), \hat{\phi}(x',t)\right] = 0$$

$$\left[\hat{\pi}(x,t), \hat{\pi}(x',t)\right] = 0$$

analogous to:

$$[\hat{q}_a(t), \hat{p}_{a'}(t)] = i\hbar\, \delta_{aa'}$$

$$[\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$[\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

- We keep the equations of motion:

(E1) $$\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

(E2) $$\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t) \qquad \dot{\hat{p}}_a(t) = -K_a \hat{q}_a(t)$$

- Note: $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$

- Is there a Hamiltonian for 2nd quantization? Yes!

analogous to:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x,t) + \frac{1}{2}\hat{\phi}(x,t)\,(m^2-\Delta)\,\hat{\phi}(x,t)\, d^3x \qquad \hat{H} = \sum_a{}' \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

- Proposition:

With this definition of $\hat{H}$, the Heisenberg equations $i\hbar\dot{\hat{f}} = [\hat{f}, \hat{H}]$

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}] \qquad i\hbar\,\dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar\,\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}] \quad (*) \qquad i\hbar\,\dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

yield the proper eqns of motion: E1, E2.

Indeed, e.g.:

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), H] = \left[\hat{\phi}(x,t), \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \text{something}(\hat{\phi})\, d^3x'\right]$$

$$= \frac{1}{2}\int [\hat{\phi}(x,t), \hat{\pi}(x',t)]\,\hat{\pi}(x',t) + \hat{\pi}(x',t)\,[\hat{\phi}(x,t), \hat{\pi}(x',t)]\, d^3x'$$

$$= \frac{i\hbar}{2}\int \delta^3(x-x')\,\hat{\pi}(x',t) + \hat{\pi}(x',t)\,\delta^3(x-x')\, d^3x' = \hat{\pi}(x,t)\, i\hbar \checkmark$$

Exercise: Prove (*)