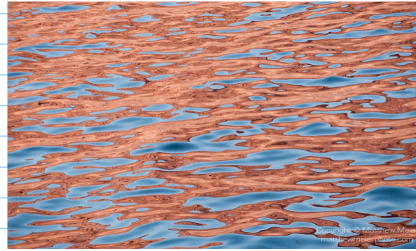


From Heisenberg to Schrödinger picture

Water:

$$\phi(x, t)$$



Probe amplitudes,
e.g., with a cork:



Quantum field:

$$\hat{\phi}(x, t)$$

How to
visualize an
operator-valued
field ?

Probe amplitudes, e.g.,
with atoms (lecture 8):



For now...

Assume we have some means to measure

$$\hat{\phi}(x, t)$$

at a time t for all $x \in \mathbb{R}^3$.

Q: Why possible in principle?

A: Because $\hat{\phi}^\dagger(x, t) = \hat{\phi}(x, t)$ and $[\hat{\phi}(x, t), \hat{\phi}(x', t)] = 0 \quad \forall x, x' \in \mathbb{R}^3$

Note: The $\hat{\phi}(x, t) \quad \forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.

⇒ At each x obtain real-valued measurement outcome, $\phi(x)$.
 Analogous to measuring \hat{q}_a and obtaining measurement outcomes q_a .

Definition: Assume that $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}$ is an arbitrary function.
 Then we define

$$|\phi\rangle \in \mathcal{X}$$

to be the joint eigenvector of all $\hat{\phi}(x,t)$ obeying
 unique up to a phase \uparrow $\hat{\phi}(x,t)$ \uparrow i.e. for all $x \in \mathbb{R}^3$

$$\hat{\phi}(x,t)|\phi\rangle = \phi(x)|\phi\rangle \quad \text{for all } x \in \mathbb{R}^3$$

Analogous to: $\hat{q}_a(t)|q\rangle = q_a|q\rangle$ for all $a = 1, \dots, 3N$
 \downarrow \times of particles

Example: Assume system is in the vacuum state $|\Omega\rangle$.

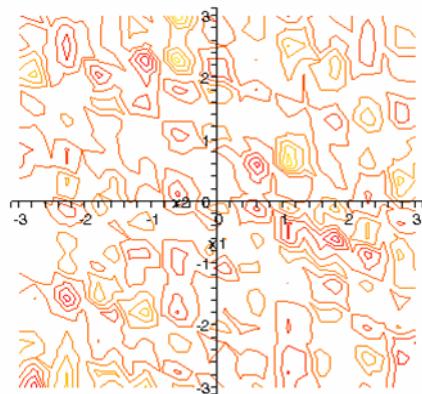
What is then a typical measurement outcome $\phi(x)$?

Shown are the level curves of a typical measurement outcome $\phi(x)$.

The measurement collapses the system into the new state:

$$|\phi\rangle \in \mathcal{X}$$

Analogous to a state $|q\rangle = |q_1, q_2, q_3, \dots\rangle$



$|\phi\rangle$ is a joint eigenstate of all $\hat{\phi}(x,t)$: $\hat{\phi}(x,t)|\phi\rangle = \phi(x)|\phi\rangle$

How to calculate in the Schrödinger picture?

Preparations:

Hilbert basis: The set $\{|\phi\rangle\}$

of all joint eigenvectors of the $\hat{p}(x,t)$ for all $x \in \mathbb{R}^3$ can be used to form a "complete ON basis" of \mathcal{H} . (up to functional analytic subtleties).

Resolution of the identity:

\Rightarrow For any $|\Psi\rangle \in \mathcal{H}$ we have:

$$|\Psi\rangle = \int_{\phi \in L^2(\mathbb{R}^3)} |\phi\rangle \langle \phi | \Psi \rangle D[\phi]$$

\leftarrow it's more subtle really

analogous to:

$$|\Psi\rangle = \int |q\rangle \underbrace{\langle q | \Psi \rangle}_{\text{Wave function: } \psi(q)} d^3q$$

The "Wave functional"

Recall QM:

- Assume $\{\hat{q}_i\}_{i=1}^N$ is compl. set of commuting observables, with joint eigenvectors $|q\rangle$ obeying: $\hat{q}_i |q\rangle = q_i |q\rangle$.
- Then the function Ψ , given by $\Psi(q) = \langle q | \Psi \rangle$ is called the "wave function" of $|\Psi\rangle$ in the $\{\hat{q}_i\}$ basis.

Example: $\{\hat{p}_i\}$ yield mom. wave functions $\Psi(p) = \langle p | \Psi \rangle$
 $\leftarrow p = \{p_1, p_2, \dots, p_N\}$

In QFT:

E.g., $\{\hat{\phi}(x)\}_{x \in \mathbb{R}^3}$ is compl. set of com. observables

← or, e.g., also $\in \mathcal{H}(\hat{\pi}(x))$.

with joint eigenvectors $|\phi\rangle$ obeying $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$.

□ Then, Ψ , given by

$$\Psi[\phi] := \langle \phi | \Psi \rangle$$

(Convention: square bracket because argument is a function)

(called a "functional" because argument is a function)

$\{|\phi\rangle\}$ form field ON eigen basis

alternatively could use e.g. joint eigen basis of the $\hat{\pi}(x,t)$.

is called the "wave functional".

Interpretation of $\Psi[\phi]$?

□ Assume the system is in an arbitrary state $|\Psi\rangle \in \mathcal{X}$ at t .

e.g., vacuum $|\Psi_0\rangle$

□ If measuring now $\hat{\phi}(x,t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $\phi(x)$?

Answer: $\text{prob}[|\Psi\rangle \rightarrow |\phi\rangle] = |\langle \phi | \Psi \rangle|^2 = |\Psi[\phi]|^2$

Q: The eqn. of motion for $\Psi[\phi, t]$?

A: The QFT Schrödinger equation!

□ For every quantum theory, we have in the Schrödinger picture of the time evolution:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

□ Which form does it take for $\Psi[\phi, t]$?

□ Here in QFT:

$$\hat{H} = \int \frac{1}{2} \left(\hat{\pi}^2(x) + \hat{\phi}(x) (-\Delta + m^2) \hat{\phi}(x) \right) d^3x$$

now independent of time!

□ But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wavefunctionals $\Psi[\phi, t]$?

□ A valid representation of $[\hat{\phi}(x), \hat{\pi}(x)] = i\delta^3(x-x')$ is: (Exercise: check)

$$\hat{\phi}(x) \cdot \Psi[\phi, t] = \phi(x) \Psi[\phi, t] \quad \text{Analogous to: } \hat{q}_a \cdot \Psi(q, t) = q_a \Psi(q, t)$$

$$\hat{\pi}(x) \cdot \Psi[\phi, t] = -i \frac{\delta}{\delta \phi(x)} \Psi[\phi, t] \quad \hat{p}_a \cdot \Psi(q, t) = -i \frac{\partial}{\partial q_a} \Psi(q, t)$$

□ Therefore: functional derivative, as in variational principle used to derive Euler Lagrange equations.

$$\hat{H} = \int \frac{1}{2} \left(-\frac{\delta^2}{\delta \phi^2(x)} + \phi(x) (-\Delta + m^2) \phi(x) \right) d^3x$$

inconvenient

□ It is more convenient to use infrared-regularized momentum space:

□ We now need to represent

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = i\delta_{k, -k'}$$

on the wave functionals $\Psi[\tilde{\phi}, t]$.

($\tilde{\phi}_k$ is Fourier transform of $\phi(x)$)

□ As you should verify, this works:

$$\hat{\phi}_k \cdot \Psi[\tilde{\phi}, t] = \tilde{\phi}_k \Psi[\tilde{\phi}, t]$$

$$\hat{\pi}_k \cdot \Psi[\tilde{\phi}, t] = -i \frac{\partial}{\partial \tilde{\phi}_{-k}} \Psi[\tilde{\phi}, t]$$

Note: Ordinary derivatives here because set of variables $\{\tilde{\phi}_k\}$ is discrete, since $k = \frac{2\pi}{L}(n_1, n_2, n_3)$, $\vec{n} \in \mathbb{Z}^3$.

⇒ Schrödinger equation:

$i\partial_t |\psi\rangle = \hat{H} |\psi\rangle$ becomes:

$$i\partial_t \Psi[\tilde{\phi}, t] = \sum_k \frac{1}{2} \left(-\frac{\partial}{\partial \tilde{\phi}_k} \frac{\partial}{\partial \tilde{\phi}_{-k}} + (k^2 + m^2) \tilde{\phi}_k \tilde{\phi}_{-k} \right) \Psi[\tilde{\phi}, t]$$

Recall: For QM harm. osc., ground state Schrödinger wave function is:

$$\Psi(x, t) = N e^{-\frac{1}{2}\omega x^2 - i\omega t}$$

Exercise: check it. Can you solve for excited states?

⇒ Ground state solution in QFT reads, similarly:

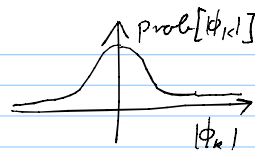
$$\Psi[\tilde{\phi}, t] = N e^{-\sum_k \left(\frac{1}{2} \omega_k \tilde{\phi}_k \tilde{\phi}_{-k} - i\omega_k t \right)}$$

Exercise: verify

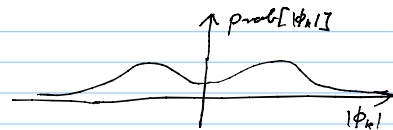
... which we had already seen before.

Generic wave functionals

- Assume the system is in a state, $|d\rangle$, other than $|\psi_0\rangle$.
 - ⇒ For at least some modes oscillators, $|d\rangle$ is not the ground state.
- But if an oscillator is excited, then its wave function spreads out - classically its amplitude of oscillation would increase.
 - ⇒ If a mode k is excited then the prob. distribution of the ϕ_k spreads:



ground state



example of excited state

- The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a ϕ_k with large modulus $|\phi_k|$.
 - ⇒ If, e.g., a mode k is very highly excited then $|\phi_k|$ is likely very large,
 - ⇒ measurement of $\hat{\phi}(x)$ likely yields $\phi(x)$ which shows a plane wave in the direction k - on top of the usual quantum fluctuations.
 - ⇒ see project description.