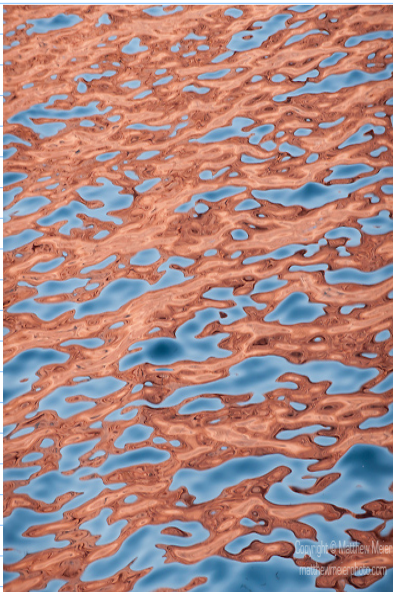


Particles in QFT

Back in the Heisenberg picture, to solve the QFT is to solve:



- The hermiticity conditions:

$$\hat{\phi}^+(x, t) = \hat{\phi}(x, t), \quad \hat{\pi}^+(x, t) = \hat{\pi}(x, t)$$

- The canonical commutation relations:

$$[\hat{\phi}(x, t), \hat{\pi}(x', t)] = i \delta(x - x')$$

- The equations of motion:

$$\hat{\pi}^{\dot{}}(x, t) - \Delta \hat{\phi}(x, t) + m^2 \hat{\phi}(x, t) = 0$$

$$\hat{\pi}(x, t) = \hat{\phi}^{\dot{}}(x, t)$$

To simplify:

- Infrared regularization:

Box size $L \times L \times L$ with periodic boundary conditions.

↑ Project: uses Dirichlet boundary conditions.

- Then Fourier series expansion:

$$\hat{\phi}(x, t) = L^{-3/2} \sum_k \hat{\phi}_k(t) e^{ikx}$$

↖ $k = \frac{2\pi}{L}(n_1, n_2, n_3), n_i \in \mathbb{Z}$

Obtain:

$$\hat{\phi}_k^{\ddot{}}(t) = -(k^2 + m^2) \hat{\phi}_k(t) \quad \text{and} \quad [\hat{\phi}_k, \hat{\phi}_{k'}] = i \delta_{k, -k'}$$

$$\hat{H} = \sum_k \hat{H}_k \quad \text{with} \quad \hat{H}_k = \frac{1}{2} \hat{\pi}_k^{\dagger} \hat{\pi}_k + \frac{1}{2} \hat{\phi}_k^{\dagger} (k^2 + m^2) \hat{\phi}_k$$

i.e.:

$$\hat{H} = \sum_k \left(\frac{1}{2} \hat{\pi}_k^{\dagger}(t) \hat{\pi}_k(t) + \frac{1}{2} \hat{\phi}_k^{\dagger}(t) (k^2 + m^2) \hat{\phi}_k(t) \right)$$

□ Crucial observations:

* For each wave vector $k = (k_x, k_y, k_z)$ there is an independent harmonic oscillator with frequency $\omega_k = \sqrt{k^2 + m^2}$ and spectrum $\text{spec}(H_k) = \hbar\omega_k \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\}$.

⇒ The excitation levels of H_k differ by the energy $E = \hbar\omega_k = \hbar\sqrt{k^2 + m^2}$ ($\hbar = 1$)

* This is also the energy of a particle of momentum k !

⇒ Hypothesis: Mode excitation = particle creation

Does this interpretation work?

Water:

$$\phi(x, t)$$

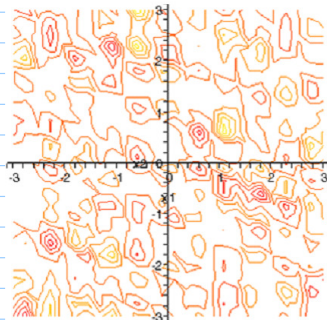


Probe amplitudes,
e.g., with a cork:



Quantum field:

$$\hat{\phi}(x, t)$$



Probe amplitudes, e.g.,
with atoms.



Use as a
detector for
the field's particles
(e.g. photons for EM field)

One finds:

- Interpretation works but is acceleration and curvature dependant.
- Interpretation simple only in Minkowski space for inertial detectors.

Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state $|\psi\rangle$ in which we have

3 particles of momentum k_a and 7 particles of momentum k_b ?

$$\begin{aligned} |\psi\rangle &= |n_{k_a}=3, n_{k_b}=7, \text{all other } n_k=0\rangle \\ &= |n_{k_a}=3\rangle \otimes |n_{k_b}=7\rangle \left(\bigotimes_{\text{all other } k_c} |n_{k_c}=0\rangle \right) \end{aligned}$$

$$\text{Energy: } \hat{H}_k |\psi\rangle = \begin{pmatrix} \hbar \omega_k \left(\frac{1}{2} + 3\right) & \text{if } k=k_a \\ \hbar \omega_k \left(\frac{1}{2} + 7\right) & \text{if } k=k_b \\ \hbar \omega_k \frac{1}{2} & \text{if } k \neq k_a, k_b \end{pmatrix} |\psi\rangle$$

$$\Rightarrow \hat{H} |\psi\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k} \frac{1}{2} \omega_k \right) |\psi\rangle$$

And one can have linear combinations:

Which is, e.g., the state $|\psi\rangle$ in which we have

3 particles of momentum k_a or 7 particles of momentum k_b ,

with probability amplitudes $\alpha, \beta = \sqrt{1-\alpha^2}$?

$$|\psi\rangle = \alpha |n_{k_a}=3, \text{other } n_k=0\rangle + \beta |n_{k_b}=7, \text{other } n_k=0\rangle$$

Notice: This is not a state of fixed particle number!

Remark: Some particle species have a number conservation law, e.g., leptons, i.e. $e^-, \mu^-, \tau^-, \nu_e^-, \nu_\mu^-, \nu_\tau^-$ (where the antiparticles count negatively).

"Superselection rule" then says we can't have such linear combinations: only number eigenstates allowed.

Mechanisms for mode excitation/particle creation?

J.e.: What are mechanisms for exciting harmonic oscillators?

□ 2 types of mechanism: (here, $\hat{q}(t)$ stands for $\hat{\phi}_k(t)$)

we'll begin with this effect → a.) A "driving force" shakes the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{j}(t)$$

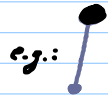
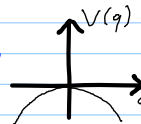


b.) A time dependence of ω affects the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2(t) \hat{q}(t)$$



And, $\omega^2(t)$ could even go negative!



All occur in QFT:

a.) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with \hat{j} terms, e.g.:

$$H(\hat{\phi}, \hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda \hat{\phi}(x,t) \hat{\psi}(x,t) d^3x$$

□ Wave interpretation: Nontrivial interaction of waves of different types of fields

□ Particle interpretation: The collision of particles happens when their mode oscillators drive another.
→ Collisions can create and annihilate particles.

□ Strongest effects?

When oscillator "resonates" with driving force.

E.g.: It takes high energy particles to make high energy particles

b.) The presence of gravity can effectively influence the $\omega_k(t)$.

▢ Wave interpretation: * E.g., cosmic expansion stretches the wavelength

⇒ expect $\omega = \omega(t)$ decreases. True, and also:

* if wavelength $>$ horizon then $\omega^2(t) < 0!$

⇒ runaway harmonic mode "oscillators"



(then: field amplification but no particle interpretation)

▢ Particle interpretation:

Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

▢ Strongest effects? When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

Case a: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

▢ We model the electromagnetic field as a Klein Gordon field.

(The fact that EM fields have polarization and have $m=0$ is not important here)

▢ Consider an arbitrary mode of the electromagnetic field:

$$\hat{\phi}_k(t)$$

should really be quantized too



▢ We model the electric current as a given classical field $\mathbf{j}(x,t)$ whose modes are $\mathbf{j}_k(t)$.

↙ should really be vector-valued

□ In a rough simplification, the EM k mode obeys:

$$\hat{H}_k = \frac{1}{2} \hat{\pi}_k^\dagger(t) \hat{\pi}_k(t) + \frac{1}{2} \omega_k^2 \hat{\phi}_k^\dagger(t) \hat{\phi}_k(t) + \hat{\phi}_k^\dagger(t) j_k(t)$$

⇒ $\int j(t)$ the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

⇒ Need to study the quantized driven harmonic oscillator!

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - j(t) \hat{q}(t)$$

Annotations for the equation above:

- for $\hat{H}_k(t)$ (points to the first term)
- for $\hat{\pi}_k(t)$ (points to the second term)
- stands for a field mode $\hat{\phi}_k(t)$ (points to the third term)
- stands for a mode $j_k(t)$ of another classical (or better quantum) field. (points to the fourth term)

I Preparation:

□ Recall that for all observables \hat{f} :

$$\bar{f}(t) = \langle \chi_0 | \hat{U}^\dagger(t) \hat{f}_0 \hat{U}(t) | \chi_0 \rangle$$

Annotations for the equation above:

- state at initial time (points to $|\chi_0\rangle$)
- operator at initial time (points to \hat{f}_0)

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \quad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

Annotations for the equation above:

- the original Hamiltonian (points to $\hat{H}(t)$)
- "Heisenberg Hamiltonian" (points to $\hat{H}(t)$)

□ Schrödinger picture? We write, equivalently: Exercise: check!

$$\begin{aligned} \bar{f}(t) &= \langle \chi_0 | \hat{U}^\dagger(t) \hat{f}_0 \underbrace{(\hat{U}(t) | \chi_0 \rangle)}_{= |\chi(t)\rangle} \\ &= \langle \chi(t) | \hat{f}_0 | \chi(t) \rangle \end{aligned}$$

Recall: $\hat{A}_S(t) = \hat{A}(t)$ only if $\frac{d}{dt} \hat{A}(t) = 0$

The dynamics is $i \frac{d}{dt} |\psi(t)\rangle = \hat{H}_S(t) |\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_S(t) = \hat{U}(t) \hat{H}(t) \hat{U}^\dagger(t)$

Exercise: check

□ We will use, equivalently, the Heisenberg picture:

$$\begin{aligned} \bar{f}(t) &= \langle \psi_0 | \underbrace{(\hat{U}^\dagger(t) \hat{f}_0 \hat{U}(t))}_{\hat{f}(t)} | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{f}(t) | \psi_0 \rangle \end{aligned}$$

with dynamics:

$$i \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say $|\psi\rangle \in \mathcal{H}$ (from measurement to measurement).
- The observables, say $\hat{H}(t)$, $\hat{f}(t)$, etc, are time-dependent operators in Hilbert space.
- Important implication:

The eigenbases and the eigenvalues of observables, such as $\hat{H}(t)$ and any $\hat{f}(t)$ depend on time!

$$\hat{f}(t) |f_n(t)\rangle = f_n(t) |f_n(t)\rangle$$

$$\hat{H}(t) |E_n(t)\rangle = E_n(t) |E_n(t)\rangle$$

Example: * Assume the driven harmonic oscillator starts out at time t_1 in n 'th energy state, say $|\psi\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t_1)|E_n(t_1)\rangle = E_n(t_1)|E_n(t_1)\rangle$$

* State vector of the system stays $|\psi\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$, the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t)|E_n(t)\rangle = E_n(t)|E_n(t)\rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1), |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

\Rightarrow At time t_2 system is still in state $|\psi\rangle$ and still

$$|\psi\rangle = |E_n(t_1)\rangle$$

but $|\psi\rangle$ is generally no longer n 'th (or any other) energy eigenstate!

In particular:

* Assume system starts out at t_1 in lowest energy state (i.e. in vacuum): $|\psi\rangle = |E_0(t_1)\rangle$

* Then if $|\psi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

\Rightarrow At t_2 the system's state $|\psi\rangle$ is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time t_2 .

III Strategy for solving quantized driven harmonic oscillator

▢ Problem: * CCR: $[\hat{q}(t), \hat{p}(t)] = i1$

* Hermiticity: $\hat{q}^\dagger(t) = \hat{q}(t), \hat{p}^\dagger(t) = \hat{p}(t)$

* Hamiltonian: $\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$

* Heisenberg eqns $i \dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ yield:

$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2 \hat{q}(t) + J(t)$$

This is a good strategy
with and without a
driving force

▢ Strategy: * Combine

$a(t) := \alpha \hat{q}(t) + i\beta \hat{p}(t)$ (analogous to "real" & "imaginary" parts)
↓ is operator even though no "hat"
↑ Mukhanov calls it $a^-(t)$

* Choose α, β so that $\hat{H}(t)$ and eqn of motion simplify.

IV Determine α and β :

▢ Notice first that once we have $a(t)$ we immediately obtain $\hat{q}(t), \hat{p}(t)$: Use of $a^\dagger(t) = \alpha \hat{q}(t) - i\beta \hat{p}(t)$ yields:

$$\hat{q}(t) = \frac{1}{2\alpha} (a^\dagger(t) + a(t))$$

$$\hat{p}(t) = \frac{i}{2\beta} (a^\dagger(t) - a(t))$$

▢ Use this to express $[\hat{q}, \hat{p}] = i$ in terms of new variable $a(t)$:

$$\Rightarrow [a(t), a^\dagger(t)] = 2\alpha\beta$$

For simplicity, we choose $\beta = \frac{1}{2\alpha}$ so that:

$$[a(t), a^\dagger(t)] = 1$$

□ Now express $\hat{H}(t)$ in terms of new variable $a(t)$:

$$\hat{H}(t) = -\frac{1}{2}d^2 (a^+(t) - a(t))^2 + \frac{\omega^2}{2} \frac{1}{4d^2} (a^+(t) + a(t))^2 - J(t) \frac{1}{2d} (a^+(t) + a(t))$$

We notice that the terms $\sim a^+(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$-\frac{1}{2}d^2 + \frac{\omega^2}{2} \frac{1}{4d^2} = 0$$

Thus, we choose: $d = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

□ Thus, by definition:

$$a(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$

□ The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega (a^+(t)a(t) + \frac{1}{2}) - J(t) \frac{1}{\sqrt{2\omega}} (a^+(t) + a(t))$$

IV Solve for $a(t)$:

□ The Heisenberg equation $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ reads for $a(t)$:

$$i \dot{a}(t) = \omega a(t) - \frac{i}{\sqrt{2\omega}} J(t)$$

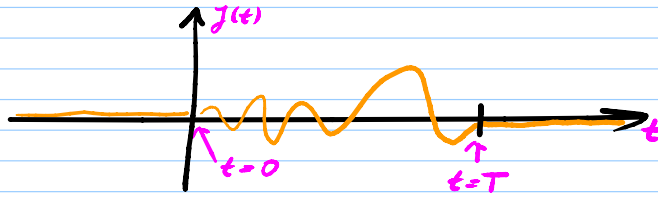
□ Let us give $a(t=0)$ a name: $a_{in} = a(0)$. Then:

Exercise:
verify.

$$a(t) = a_{in} e^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

VI Case of force of finite duration

□ Assume $J(t) = 0$ for all $t \notin [0, T]$



□ Define $J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

□ Then:
$$a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_{in} + J_0) e^{-i\omega t} & \text{for } t > T \end{cases}$$

Implications in terms of particle (e.g. photon) production?