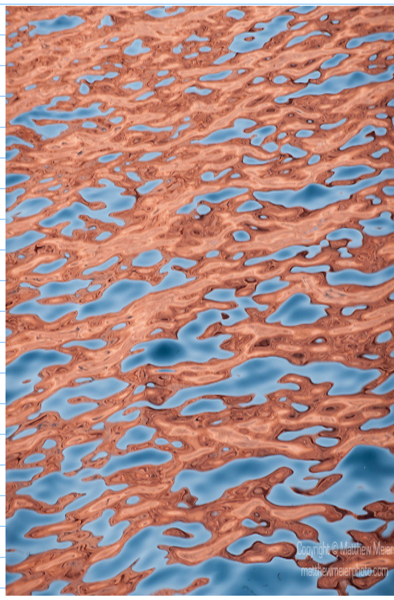
QFT for Cosmology, Achim Kempf, Lecture 5

Particles in QFT

Back in the Heisenberg picture, to solve the QFT is to solve:



- The hermiticity conditions:

$$\hat{\phi}^+(x,t) = \hat{\phi}(x,t) \quad , \quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t)$$

- The canonical commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\,\delta(x-x')$$

- The equations of motion:

$$\dot{\hat{\pi}}(x,t) - \Delta\hat{\phi}(x,t) + m^2\hat{\phi}(x,t) = 0$$

$$\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$$

To simplify:

- Infrared regularization:

  Box size $L \times L \times L$ with periodic boundary conditions.

  Project: uses Dirichlet boundary conditions.

- Then Fourier series expansion:

$$\hat{\phi}(x,t) = L^{-3/2} \sum_k \hat{\phi}_k(t)\, e^{ikx}$$

$$k = \tfrac{2\pi}{L}(n_1, n_2, n_3),\ n_i \in \mathbb{Z}$$

Obtain:

$$\ddot{\hat{\phi}}_k(t) = -(k^2+m^2)\,\hat{\phi}_k(t) \quad \text{and} \quad [\hat{\phi}_k, \dot{\hat{\phi}}_{k'}] = i\,\delta_{k,-k'}$$

$$\hat{H} = \sum_k \hat{H}_k \quad \text{with} \quad \hat{H}_k = \tfrac{1}{2}\hat{\pi}_k^+\hat{\pi}_k + \tfrac{1}{2}\hat{\phi}_k^+(k^2+m^2)\hat{\phi}_k$$

i.e.:

$$\hat{H} = \sum_k \left(\tfrac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \tfrac{1}{2}\hat{\phi}_k^+(t)\,(k^2+m^2)\,\hat{\phi}_k(t) \right)$$

Crucial observations:

* For each wave vector $k=(k_1,k_2,k_3)$ there is an independent harmonic oscillator with frequency $\omega_k=\sqrt{k^2+m^2}$ and spectrum $spec(H_k)=\hbar\omega_k\left\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots\right\}$.

$\Rightarrow$ The excitation levels of H_k differ by the energy

$$E=\omega_k=\sqrt{k^2+m^2} \qquad (\hbar=1)$$

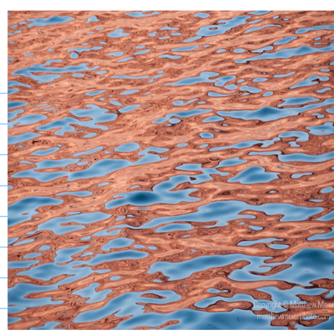
* This is also the energy of a particle of momentum k!

$\rightsquigarrow$ Hyothesis: $\boxed{\text{Mode excitation = particle creation}}$

Does this interpretation work?

Water:

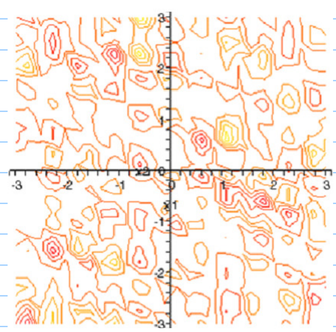
$\phi(x,t)$



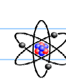
Probe amplitudes, e.g., with a cork:

Quantum field:

$\hat{\phi}(x,t)$



Probe amplitudes, e.g., with atoms.



Use as a detector for the field's particles (e.g. photons for EM field)

One finds:

- Interpretation works but is acceleration and curvature dependent.
- Interpretation simple only in Minkowski space for inertial detectors.

Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state $|\psi\rangle$ in which we have

3 particles of momentum k_a and 7 particles of momentum k_b?

$$|\psi\rangle = |n_{k_a}=3,\, n_{k_b}=7,\, \text{all other } n_k=0\rangle$$
$$= |n_{k_a}=3\rangle \otimes |n_{k_b}=7\rangle \left(\bigotimes_{\substack{\text{all other}\\ k_c}} |n_{k_c}=0\rangle \right)$$

Energy:
$$\hat{H}_k|\psi\rangle = \begin{pmatrix} \hbar\omega_k\left(\frac{1}{2}+3\right) & \text{if } k=k_a \\ \hbar\omega_k\left(\frac{1}{2}+7\right) & \text{if } k=k_b \\ \hbar\omega_k\frac{1}{2} & \text{if } k\neq k_a, k_b \end{pmatrix}|\psi\rangle$$

$$\Rightarrow \quad \hat{H}|\psi\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k}\frac{1}{2}\omega_k\right)|\psi\rangle$$

And one can have linear combinations:

Which is, e.g., the state $|x\rangle$ in which we have

3 particles of momentum k_a or 7 particles of momentum k_b,

with probability amplitudes α, $\beta = \sqrt{1-|\alpha|^2}$?

$$|x\rangle = \alpha\,|n_{k_a}=3, \text{other } n_k=0\rangle + \beta\,|n_{k_b}=7, \text{other } n_k=0\rangle$$

Notice: This is not a state of fixed particle number!

(Remark: Some particle species have a number conservation law, e.g., leptons, i.e. $e^-, \mu^-, \tau^-, \nu_{e^-}, \nu_{\mu^-}, \nu_{\tau^-}$, (where the antiparticles count negatively). "Superselection rule" then says we can't have such linear combinations: only number eigenstates allowed.)

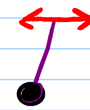
Mechanisms for mode excitation / particle creation?

I.e.: What are mechanisms for exciting harmonic oscillators?

- **2 types of mechanism:** (here, $\hat{q}(t)$ stands for $\hat{\phi}_k(t)$)

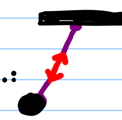
we'll begin with this effect → **a)** A "driving force" shakes the oscillator:

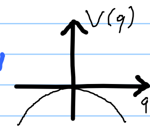
$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{J}(t)$$

e.g.: 

b) A time dependence of ω affects the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2(t)\, \hat{q}(t)$$

e.g.: 

And, $\omega^2(t)$ could even go negative! 

e.g.: 

All occur in QFT:

A) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with J terms, e.g.:

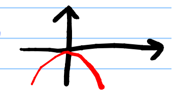
$$H(\hat{\phi}, \hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda\, \hat{\phi}(x,t)\, \hat{\psi}(x,t)\, d^3x$$

- **Wave interpretation:** Nontrivial interaction of waves of different types of fields
- **Particle interpretation:** The collision of particles happens when their mode oscillators drive one other.
  → Collissions can create and annihilate particles.
- **Strongest effects?**

  When oscillator "resonates" with driving force.
  E.g.: It takes high energy particles to make high energy particles

B) The presence of gravity can effectively influence the $\omega_k(t)$.

- Wave interpretation: * E.g., cosmic expansion stretches the wavelength
  $\Rightarrow$ expect $\omega = \omega(t)$ decreases. True, and also:
  * if wavelength > horizon then $\omega^2(t) < 0$!
  $\Rightarrow$ runaway harmonic mode "oscillators"



  (then: field amplification but no particle interpretation)

- Particle interpretation:

  Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

- Strongest effects? When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

Case A: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

- We model the electromagnetic field as a Klein Gordon field.

  (The fact that EM fields have polarization and have $m=0$ is not important here)

- Consider an arbitrary mode of the electromagnetic field:

$$\hat{\phi}_k(t)$$

should really be quantized too

- We model the electric current as a given classical field $j(x,t)$ whose modes are $j_k(t)$.

  should really be vector-valued

- In a rough simplification, the EM k mode obeys:

$$\hat{H}_k = \frac{1}{2}\hat{\pi}_k^+(t)\,\hat{\pi}_k(t) + \frac{1}{2}\omega_k^2\,\hat{\phi}_k^+(t)\,\hat{\phi}_k(t) + \hat{\phi}_k(t)\,j_k(t)$$

$\Rightarrow$ If the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

$\Rightarrow$ Need to study the quantized driven harmonic oscillator!

for $\hat{H}_k(t)$ — for $\hat{\pi}_k(t)$ — stands for a field mode $\hat{\phi}_k(t)$

$$\boxed{\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)}$$

stands for a mode $j_k(t)$ of another classical (or better quantum) field.

I Preparation:

- Recall that for all observables $\hat{f}$:

state at initial time

$$\bar{f}(t) = \langle\psi_0|\,\hat{U}^+(t)\,\hat{f}_0\,\hat{U}(t)\,|\psi_0\rangle$$

operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \qquad i\frac{d}{dt}\hat{U}(t) = \hat{U}(t)\,\hat{H}(t)$$

"Heisenberg Hamiltonian" — the original Hamiltonian

- Schrödinger picture? We write, equivalently: (Exercise: check!)

$$\bar{f}(t) = \left(\langle\psi_0|\,\hat{U}^+(t)\right)\hat{f}_0\left(\overbrace{\hat{U}(t)\,|\psi_0\rangle}^{=|\psi(t)\rangle}\right)$$

$$= \langle\psi(t)|\,\hat{f}_0\,|\psi(t)\rangle$$

Recall: $\hat{H}_s(t) = \hat{H}(t)$ only if $\frac{d}{dt}\hat{H}(t) = 0$

The dynamics is $i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_s(t)|\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_s(t) = \hat{U}(t)\hat{H}(t)\hat{U}^+(t)$

Exercise: check

- We will use, equivalently, the Heisenberg picture:

$$\bar{f}(t) = \langle\psi_0|\underbrace{\left(\hat{U}^+(t)\,\hat{f}_0\,\hat{U}(t)\right)}_{=}|\psi_0\rangle$$

$$= \langle\psi_0|\ \hat{f}(t)\ |\psi_0\rangle$$

with dynamics:

$$i\frac{d}{dt}\hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say $|\psi\rangle \in \mathcal{H}$ (from measurement to measurement).

- The observables, say $\hat{H}(t)$, $\hat{f}(t)$, etc, are time-dependent operators in Hilbert space.

- Important implication:

The eigenbases and the eigenvalues of observables, such as $\hat{H}(t)$ and any $\hat{f}(t)$ depend on time!

$$\hat{f}(t)|f_n(t)\rangle = f_n(t)\,|f_n(t)\rangle$$

$$\hat{H}(t)|E_n(t)\rangle = E_n(t)\,|E_n(t)\rangle$$

Example: * Assume the driven harmonic oscillator starts out at time t_1 in n'th energy state, say $|\chi\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t_1)\,|E_n(t_1)\rangle = E_n(t_1)\,|E_n(t_1)\rangle$$

* State vector of the system stays $|\chi\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$ the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t)\,|E_m(t)\rangle = E_m(t)\,|E_m(t)\rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1)\ ,\ |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

$\Rightarrow$ At time t_2 system is still in state $|\chi\rangle$ and still

$$|\chi\rangle = |E_n(t_1)\rangle$$

but $|\chi\rangle$ is generally no longer n'th (or any other) energy eigenstate!

In particular:

* Assume system starts out at t_1 in lowest energy state (i.e., in vacuum): $|\chi\rangle = |E_0(t_1)\rangle$

* Then if $|\chi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

$\Rightarrow$ At t_2 the system's state $|\chi\rangle$ is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time t_2.

III Strategy for solving quantized driven harmonic oscillator

Problem:

* CCR: $[\hat{q}(t), \hat{p}(t)] = i\mathbb{1}$
* Hermiticity: $\hat{q}^+(t) = \hat{q}(t)$, $\quad \hat{p}^+(t) = \hat{p}(t)$
* Hamiltonian: $\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\hat{q}(t)$
* Heisenberg eqns $\quad i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ yield:

$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2\hat{q}(t) + J(t)$$

This is a good strategy with and without a driving force

Strategy:

* Combine

is operator even though no "hat".

$$a(t) := \alpha\,\hat{q}(t) + i\beta\,\hat{p}(t)$$

Mukhanov calls it $a^-(t)$

(analogous to "real" & "imaginary" parts)

* Choose α, β so that $\hat{H}(t)$ and eqn of motion simplify.

IV Determine α and β:

Notice first that once we have $a(t)$ we immediately obtain $\hat{q}(t), \hat{p}(t)$: Use of $a^+(t) = \alpha\,\hat{q}(t) - i\beta\,\hat{p}(t)$ yields:

$$\hat{q}(t) = \frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

$$\hat{p}(t) = \frac{i}{2\beta}\left(a^+(t) - a(t)\right)$$

Use this to express $[\hat{q}, \hat{p}] = i$ in terms of new variable $a(t)$:

$$\Rightarrow \qquad [a(t), a^+(t)] = 2\alpha\beta$$

For simplicity, we choose $\beta = \frac{1}{2\alpha}$ so that:

$$\boxed{[a(t), a^+(t)] = 1}$$

- Now express $\hat{H}(t)$ in terms of new variable $a(t)$:

$$\hat{H}(t) = -\frac{1}{2}\alpha^2 \left(a^+(t) - a(t)\right)^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2}\left(a^+(t)+a(t)\right)^2$$

$$- J(t)\frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

We notice that the terms $\sim a^+(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$-\frac{1}{2}\alpha^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2} = 0$$

Thus, we choose: $\alpha = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

- Thus, by definition:

$$\boxed{a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)}$$

- The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega\left(a^+(t)a(t) + \frac{1}{2}\right) - J(t)\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)$$

V Solve for $a(t)$:

- The Heisenberg equation $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ reads for $a(t)$:

$$i\,\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}}J(t)$$

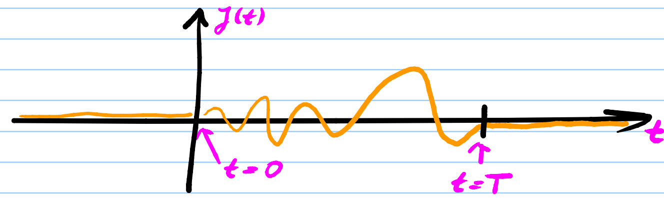
- Let us give $a(t=0)$ a name: $a_{in} = a(0)$. Then:

Exercise: verify.

$$\boxed{a(t) = a_{in}\, e^{-i\omega t} + 1\,\frac{i}{\sqrt{2\omega}}\int_0^t J(t')\, e^{i\omega(t'-t)}\, dt'}$$

VI Case of force of finite duration

- Assume $J(t) = 0$ for all $t \notin [0,T]$



- Define $$J_0 := \frac{i}{\sqrt{2w}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$$

- Then:
$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0,T] \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } t > T \end{cases}$$

Next:

Implications in terms of particle (e.g. photon) production?