Note Title

QFT for Cosmology, Achim Kempf, Lecture 6

Recall:

- There are two basic mechanisms to increase the amplitudes of oscillators, i.e., also to excite a field's mode oscillators, i.e. to create particles:

  a.) A time-varying driving force $J(t)$

  b.) A time-varying spring "constant" $\omega(t)$

- We are presently considering case a):

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$

with a temporary force: $J(t) = 0$ for all $t \notin [0,T]$

Examples: 1. Temporary emission from antenna, 2. Brief interaction (scattering) of particles.

- We defined a convenient variable $a(t)$,

$$a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

so that: $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}J(t)\left(a^+(t) + a(t)\right)$

- By using $a(t)$, the problem reduced to solving:

  * $i\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}}J(t)$ (EOM)

  * $[a(t), a^+(t)] = 1$ for all t (CCR)

We gave a convenient name to $a(t=0)$:

$$a_{in} := a(t=0)$$ (an operator on Hilbert space that we still have to choose.)

Then, as is easy to verify, the solution is:

$$a(t) = a_{in} e^{-i\omega t} + 1 \frac{i}{\sqrt{2\omega}} e^{-i\omega t} \int_0^t J(t') e^{i\omega t'} dt'$$

$$= \left(a_{in} + 1 \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega t'} dt' \right) e^{-i\omega t}$$

And so, with the definition: $J_0 := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

$$a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } 0 \leq t \leq T \\ \underbrace{(a_{in} + J_0)}_{!!} e^{-i\omega t} & \text{for } T < t \end{cases}$$

Define: a_{out}

Before and after the force action, we have an undriven harmonic oscillator, solved as always, by $a(t) = a_0 e^{-i\omega t}$

Here:

$$a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ a_{out} e^{-i\omega t} & \text{for } t > T \end{cases}$$

with $a_{out} = a_{in} + J_0$:

Conservation of the CCRs?

Notice that $[a_{in}, a_{in}^+] = 1$ implies $[a_{out}, a_{out}^+] = 1$.

In fact:

Proposition: If we can arrange for $[a_{in}, a^+_{in}] = 1$,

then $[a(t), a^+(t)] = 1$ follows for all $t \in \mathbb{R}$!

Proof:

Assume $[a_{in}, a^+_{in}] = 1$. Then:

$$[a(t), a^+(t)] = [a_{in} e^{-i\omega t} + \underbrace{1 \frac{i}{\sqrt{2\omega}} \int_0^t \ldots dt'}_{\text{number}}, a^+_{in} e^{+i\omega t} - \underbrace{1 \frac{i}{\sqrt{2\omega}} \int_0^t \ldots dt'}_{\text{number}}]$$

$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad \checkmark$$

The initial period, $t < 0$:

▫ The dynamical variables:

We have $a(t) = a_{in} e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

$$* \quad \hat{q}(t) = \frac{1}{\sqrt{2\omega}} \left(a^+_{in} e^{i\omega t} + a_{in} e^{-i\omega t} \right)$$

$$* \quad \hat{p}(t) = i\sqrt{\frac{\omega}{2}} \left(a^+_{in} e^{i\omega t} - a_{in} e^{-i\omega t} \right)$$

Exercise: verify

$$* \quad \hat{H}(t) = \omega \left(a^+(t) a(t) + \frac{1}{2} \right)$$

$$= \omega \left(a^+_{in} e^{i\omega t} a_{in} e^{-i\omega t} + \frac{1}{2} \right)$$

$$= \omega \left(a^+_{in} a_{in} + \frac{1}{2} \right)$$ is constant in time!

The Hilbert space of states:

* As always, we can write arbitrary Hilbert space vectors as linear combinations of an arbitrary set of basis vectors.

* We could use, for example, the eigenbasis of $\hat{q}(t)$ (or the eigenbasis of $\hat{p}(t)$).

  But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigen basis for each t.

* However, $\hat{H}$ is time independent (for $t<0$).

  ⇝ Let us construct and use its eigenbasis:

The eigenbasis of $\hat{H}$ for $t<0$:

* We have

$$\hat{H}_{t<0} = \omega\left(a^+_{in}\, a_{in} + \tfrac{1}{2}\right)$$

with:

$$[a_{in}, a^+_{in}] = 1 \qquad \text{(CCR)}$$

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in}\,|0_{in}\rangle = 0$$

the Hilbert space vector with zero length

* Then it is eigenvector of $H_{t<0}$:

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$ i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar = 1$).

$$\hat{H}_{t<0}|0_{in}\rangle = \omega\left(a^+_{in} a_{in} + \tfrac{1}{2}\right)|0_{in}\rangle = \tfrac{1}{2}\,\omega\,|0_{in}\rangle$$

⇒ We recognize $|0\rangle$: it is the lowest energy eigenvector of $\hat{H}$ (and thus it indeed exists)

* Consider now the state $|1_{\underline{m}}\rangle := a^+_{\underline{m}}|0_{\underline{m}}\rangle$:

$$\hat{H}_{t<0}|1_{\underline{m}}\rangle = \hat{H}_{t<0}a^+_{\underline{m}}|0_{\underline{m}}\rangle = \omega(a^+_{\underline{m}}a_{\underline{m}}+\tfrac{1}{2})a^+_{\underline{m}}|0_{\underline{m}}\rangle$$

$$= \left(\omega a^+_{\underline{m}}(a^+_{\underline{m}}\cancel{a_{\underline{m}}}+1)+\tfrac{\omega}{2}a^+_{\underline{m}}\right)|0_{\underline{m}}\rangle$$

$$= \omega\,\tfrac{3}{2}\,a^+|0_{\underline{m}}\rangle$$

$$= \tfrac{3}{2}\omega\,|1_{\underline{m}}\rangle$$

$\Rightarrow$ The state $|1_{\underline{m}}\rangle$ is eigenstate of $\hat{H}$ with eigenvalue $\frac{3}{2}\omega$. So it must be the 1^{st} excited state.

* Is the vector $|1_{\underline{m}}\rangle$ normalized?

$$\langle 1_{\underline{m}}|1_{\underline{m}}\rangle = \langle 0_{\underline{m}}|a_{\underline{m}}a^+_{\underline{m}}|0_{\underline{m}}\rangle = \langle 0_{\underline{m}}|a^+_{\underline{m}}a_{\underline{m}}+1|0_{\underline{m}}\rangle = \langle 0_{\underline{m}}|0_{\underline{m}}\rangle = 1 \quad ✓$$

* **Proposition:**

The set of vectors $\{|n_{\underline{m}}\rangle\}_{n=0}^{\infty}$ defined through

$$|n_{\underline{m}}\rangle := \frac{1}{\sqrt{n!}}(a^+_{\underline{m}})^n|0_{\underline{m}}\rangle$$

is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

* **Proposition:**

The vectors $|n_{\underline{m}}\rangle$ are eigenvectors of $\hat{H}_{t<0}$:

$$\left.\begin{aligned} \hat{H}_{t<0}|n_{\underline{m}}\rangle &= E_n|n_{\underline{m}}\rangle \\ \text{with}\quad E_n &= \omega(n+\tfrac{1}{2}) \end{aligned}\right\}\ \text{Exercise: verify}$$

* **Proposition:** $\{|n_{\underline{m}}\rangle\}$ is complete eigenbasis of $\hat{H}$.

Summary re choice of basis for $t<0$:

- The Hamiltonian $\hat{H}(t)$ is constant for $t<0$.
- Thus it has one eigenbasis for all $t<0$, namely $\{|n_{in}\rangle\}$.
- We may expand every arbitrary vector $|\psi\rangle$ of the Hilbert space, $\mathcal{H}$, in this basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n_{in}\rangle$$

- E.g., the state of our quantum system could be:

$$|\psi\rangle = |5_{in}\rangle$$

- The system always stays in state $|\psi\rangle = |5_{in}\rangle$.

Recall: ○ But $|\psi\rangle = |5_{in}\rangle$ generally ceases to be eigenvector of $\hat{H}(t)$ for $t>0$!

The period $t > T$: (after the force ceased to act)

- Once the driving force acts, $\hat{H}(t)$ starts to change.
- **But**: After the force finished, $t>T$, the Hamiltonian simply reads

$$\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{a^+(t)+a(t)}{\sqrt{2\omega}}\,\underbrace{\cancel{J(t)}}_{=0} \quad \text{for } t>T$$

and from above, therefore:

$$\hat{H}(t) = \omega\left(a^+_{out}\cancel{e^{i\omega t}}\,a_{out}\cancel{e^{-i\omega t}} + \frac{1}{2}\right) \quad \text{with } a_{out} = a_{in} + J_0$$

$$\Rightarrow \quad \hat{H}_{t>T} = \omega\left(a^+_{out}\,a_{out} + \frac{1}{2}\right) \Rightarrow \hat{H} \text{ is then constant again!}$$

- Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

Compare $t<0$ to $t>T$:

QFT:

A. Motion: $\bar{q}(t)$ (large $\bar{q}$ means large $\bar{\phi}_k$ means large waves)

B. Resonance: best $J(t)$? (consider e.g. antenna)

C. Energy expectation: $\bar{E}(t)$ (large $\bar{E}$ means large $\bar{E}_k$ means energy in mode k)

D. Energy eigenstates: $\{|E_n(t)\rangle\}$ (particle creation)

We will consider the example where the system starts out in the lowest energy state (the vacuum):

$$|\psi\rangle = |0_{in}\rangle$$

A. Motion $\bar{q}(t)$:

$$\bar{q}(t) = \langle\psi|\hat{q}(t)|\psi\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)|0_{in}\rangle$$

* For $t<0$ we obtain:

$$\bar{q}(t) = \frac{1}{\sqrt{2\omega}}\langle 0_{in}|\, a^+_{in}e^{i\omega t} + a_{in}e^{-i\omega t}\,|0_{in}\rangle$$

$$= 0$$

This was expected since for $t<0$ the system's state $|0_{in}\rangle$ is the ground state of $\hat{H}(t)$.

* For $t>T$ we obtain:

$a_{out} = a_{in} + J_0$

$$\bar{q}(t) = \langle\psi|\,\hat{q}(t)|\psi\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)|0_{in}\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}\left(a^+_{out}e^{i\omega t} + a_{out}e^{-i\omega t}\right)|0_{in}\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}\left((\cancel{a^+_{in}}+J_0^*)e^{i\omega t} + (\cancel{a_{in}}+J_0)e^{-i\omega t}\right)|0_{in}\rangle$$

$$= \frac{1}{\sqrt{2\omega}}\left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t}\right) \qquad (*)$$

Exercise: verify ↗

$$= \int_0^T \frac{\sin((t-t')\omega)}{\omega}\, J(t')\,dt'$$

(Remark: same as classical $q(t)$ due to Ehrenfest theorem)

$\Rightarrow$ $\bar{q}$ oscillates with frequency ω, as expected.

B. Resonance:

* The amplitude of the excited motion of the oscillator is determined by J_0, as equation (*) shows.

* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}}\int_0^T J(t')e^{i\omega t'}\,dt'$$

Thus, indeed, the more of the frequency ω is contained in $J(t)$, the larger is $|J_0|$.

C. Energy expectation

* For $t<0$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \qquad \text{(always)}$$

$$= \langle 0_{in} | \omega (a^+_{in} \cancel{a_{in}} + \tfrac{1}{2}) | 0_{in} \rangle \qquad (\text{for } t<0)$$

$$= \frac{\omega}{2}$$

i.e., the energy of the ground state of the Hamiltonian $\hat{H}_{t<0}$.

* For $t>T$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \qquad \text{(always)}$$

$$= \langle 0_{in} | \omega (a^+_{out} a_{out} + \tfrac{1}{2}) | 0_{in} \rangle \quad (\text{for } t>T)$$

$$= \omega \langle 0_{in} | (\cancel{a^+_{in}} + J_0^*)(\cancel{a_{in}} + J_0) + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \langle 0_{in} | J_0^* J_0 + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \left(\tfrac{1}{2} + |J_0|^2 \right) \qquad \text{which is elevated!}$$

Remark: We notice that the oscillator's energy increases the more the larger $|J_0|$, i.e., from B, the closer the driving force is to the oscillator's natural frequency ω.

Remark: In QFT, say when electrical current drives electromagnetic field modes, the closer a mode's ω_k is to the frequency of the current, the more this mode gets excited.

Implication: $|0_{out}\rangle \neq |0_{in}\rangle = |\chi\rangle$

- Ground state $|0_{out}\rangle$ of

$$H_{t>T} = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) = \omega\left(a^+_{out}\,a_{out} + \tfrac{1}{2}\right)$$

has eigenvalue $\omega/2$, i.e.:

$$a_{out}|0_{out}\rangle = 0.$$

- Therefore: $a_{out}\,|\chi\rangle = a_{out}|0_{in}\rangle$

$$= (a_{in} + J_0)\,|0_{in}\rangle$$

$$= J_0|\chi\rangle \neq 0$$

$\Rightarrow$ At late times: $|\chi\rangle \neq |0_{out}\rangle$

Q: So what kind of excited state is $|\chi\rangle$ at late times?

A: Since $|\chi\rangle$ is eigenstate of a lowering operator,

$$a_{out}\,|\chi\rangle = J_0\,|\chi\rangle$$

$|\chi\rangle$ is what is called a Coherent State.

Recall:

Coherent states saturate the uncertainty relation:

If $|\psi\rangle$ is a coherent state, then

$$\Delta q_{|\psi\rangle}\,\Delta p_{|\psi\rangle} = \frac{\hbar}{2}$$

$\rightsquigarrow$ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:

$$\hat{q}|\psi\rangle = \langle\hat{q}\rangle|\psi\rangle \text{ and } \hat{p}\,|\psi\rangle = \langle\hat{p}\rangle|\psi\rangle$$

Exercise: Show that if $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\alpha \in \mathbb{C}$

and $\hat{q} = \frac{1}{\sqrt{2\omega}}(a^+ + a)$, $\hat{p} = i\sqrt{\frac{\omega}{2}}(a^+ - a)$ $\quad (*)$

Then, $\langle\alpha|\hat{q}|\alpha\rangle = \frac{1}{\sqrt{2\omega}}(\alpha^* + \alpha)$

$$\langle\alpha|\hat{p}|\alpha\rangle = i\sqrt{\frac{\omega}{2}}(\alpha^* - \alpha)$$

and: $\Delta q(t)\, \Delta p(t) = \frac{1}{2}$

Remarks:

- Notice that because $a|\alpha\rangle = \alpha|\alpha\rangle$, the operator a does **not** reduce the excitation (or particle) number of $|\alpha\rangle$.
- If the ω in $(*)$ is chosen to be not the frequency of the harmonic oscillator of the Hamiltonian, then $|\alpha\rangle$ is called a Squeezed State.

Q: Significance of driven harmonic oscillators always ending up in a coherent state for QFT?

A: Consider example of classical currents and charges driving the mode oscillators of the electromagnetic QFT:

- The charges and currents drive the EM oscillators into a coherent state. ← (In Heisenberg picture: State stays constant but its meaning relative to the then time-dependent operators changes)
- ⇒ The $\hat{q}_k$, $\hat{p}_k$ of the EM field (essentially the $\hat{E}_i$ and $\hat{B}_i$ fields) will be as sharp as possible, i.e., the EM QFT's state is close to being eigenstate to $\hat{E}_i$ and $\hat{B}_i$.

Q: Implications for how, e.g., an electron interacts with such an EM field?

A: It explains why it is nearly legal to do what one is taught in a first course in quantum mechanics:

"In order to describe the quantum mechanics of an electron interacting with a background EM field, simply put electric and magnetic fields as number-valued functions into the Schrödinger equation."

It is indeed often a good approximation to replace the operators $\hat{E}_i(x,t)$ and $\hat{B}_i(x,t)$ by their expectation values, if the QFT is in a coherent state.

<u>Remarks:</u>

- It is also often said, when teaching quantum mechanics, that an electron's wave function will obey the Schrödinger equation (and thus evolve unitarily) only until it next interacts with some other system.

- That other system would gain information about our electron during the interaction, and this should make the electron's wave function collapse. We may not measure the other system to learn what it learned, but in any case, if the e^- started in a pure state, its interaction with the other system would decohere it.

Q: If true, why can we have a Schrödinger electron interacting strongly with another quantum system, namely the electromagnetic field, without the wave function collapsing due to the interaction?

A: Not every interaction constitutes a measurement:

- Consider the quantum systems of the e^- and the EM QFT:

$$\hat{H}^{(tot)} = \hat{H}^{(EM)} \otimes 1 + 1 \otimes \hat{H}^{(e^-)} + \hat{H}^{(int)}$$

Here, $\hat{H}^{int} = \hat{p} \otimes \hat{B}$, for example.

- Now assume the initial state of the total system, $|\psi\rangle$, is unentangled: $|\psi\rangle = |\varphi_0^{e^-}\rangle \otimes |\phi_0^{EM}\rangle$.

- In general, the time evolution operator $\hat{U}(t) = e^{-it\hat{H}^{tot}}$ will, in the Schrödinger picture, make

$$|\psi(t)\rangle = e^{-it\left(\hat{H}^{EM}\otimes 1 + 1\otimes \hat{H}^{e^-} + \hat{B}\otimes\hat{p}\right)} |\varphi_0^{e^-}\rangle \otimes |\phi_0^{EM}\rangle \quad \text{entangled.}$$

- **Why?** Because H^{tot} contains not only terms of the form $\hat{X}\otimes 1$ and $1\otimes\hat{X}$ but also of the form $X\otimes Y$, e.g., $\hat{B}\otimes\hat{p}$.

  nontrivial operators — nontrivial operators

- This means that when we trace over the EM field, the electron's state is seen to evolve from pure to mixed:

$$\rho_{e^-}(t) = Tr_{\mathcal{H}_F}\left(|\psi(t)\rangle\langle\psi(t)|\right) \text{ becomes mixed.}$$

$\mathcal{H}_F$ = Hilbert space of the EM field.

- However, if in good approximation,

$$\hat{B}_i(x)|\phi^{EM}\rangle \approx \overbrace{\underbrace{\langle \hat{B}_i(x)\rangle}_{=:\,B_i(x)}}^{\text{number-valued!}} |\phi^{EM}\rangle$$

then $\hat{H}^{tot} \approx \hat{H}^{EM} \otimes 1 + \underbrace{1 \otimes \hat{H}^{e^-} + \hat{B} \otimes \hat{p}}_{\approx B\, 1 \otimes \hat{p}}$.

number

$\Rightarrow$ $\hat{U}(t)$ is, in good approximation, not entangling.

i.e. the electron's state evolves from pure to pure, in spite of its interaction with the EM field (because it was essentially not a measurement).

Next:

Study the coherent state that a driven harmonic oscillator ends up in, for QM and then for QFT.