Note Title

Mathematical preparations for QFT in curved space:

Plan today:

- Functional derivatives $\dfrac{\delta F[g]}{\delta g(x)} = ?$
- Example use 1: to make the QFT Schrödinger equation well defined.
- Example use 2: to define the Functional Legendre transform.
- Use both to obtain the Lagrangian formulation of QFT – which will be starting point for QFT on curved space.

Functional differentiation

Recall:

a.) Differentiation of functions of one variable, $F(u)$:

$$\frac{dF(u)}{du} := \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon) - F(u)}{\varepsilon}$$

b.) Differentiation of functions of countably many variables, $F(\{u_j\}_{j=1,2,3,\dots})$:

$$\frac{\partial F(\{u_j\}_{j=1,2,\dots})}{\partial u_i} := \lim_{\varepsilon \to 0} \frac{F(u_1,\dots,u_i+\varepsilon,\dots) - F(u_1,\dots,u_i,\dots)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{F(\{u_j + \varepsilon\,\delta_{ij}\}_{j=1,\dots}) - F(\{u_j\}_{j=1,\dots})}{\varepsilon}$$

Definition:

c.) Differentiation of functions of uncountably many variables, $F(\{u(x)\}_{x\in\mathbb{R}^n})$:

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x\in\mathbb{R}^n})}{\delta u(y)} := \lim_{\varepsilon\to 0} \frac{F(\{u(x)+\varepsilon\delta^n(x-y)\}_{x\in\mathbb{R}^n}) - F(\{u(x)\}_{x\in\mathbb{R}^n})}{\varepsilon}$$

→ Since F is a "functional", i.e., is mapping functions to numbers

$$F: \underset{\substack{\uparrow \\ \text{function}}}{u} \to \underset{\substack{\uparrow \\ \text{short for } \{u(x)\}_{x\in\mathbb{R}^n}}}{F[u]} \in \mathbb{C}$$

we call $\frac{\delta F}{\delta u(x)}$ a functional derivative.

Example:

$$F[u] := \int_{\mathbb{R}} \cos(x)\, u(x)^2\, dx$$

Then:

$$\frac{\delta F}{\delta u(y)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{\mathbb{R}} \cos(x)\big(u(x)+\varepsilon\delta(x-y)\big)^2 dx - \int_{\mathbb{R}}\cos(x)\, u(x)^2 dx\right]$$

Distribution theory would be needed. But it drops out anyway

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}}\cos(x)\Big(\cancel{u(x)^2} + \varepsilon 2u(x)\delta(x-y) + \varepsilon^2\delta^2(x-y) - \cancel{u(x)^2}\Big)dx$$

$$= \lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon}\int_{\mathbb{R}} 2u(x)\,\delta(x-y)\cos(x)\, dx$$

$$= 2\cos(y)\, u(y)$$

Similarly, one obtains: $\frac{\delta}{\delta u(y)} \int_{\mathbb{R}} f(x)\, u(x)^n\, dx = f(y)\, n\, u(y)^{n-1}$

$\Rightarrow$ Functional derivatives act on polynomials (and suitable power series) in u by removing the integral and reducing the power in u by one, as expected from ordinary derivatives.

Remark:

* Worked with $u(x)$.
* Would obtain same result if we used any other continuous or discrete basis of L^2.
* E.g. other basis (continuous): e^{ixp}, i.e. use $\tilde{u}(p)$
* E.g. other basis (countable): $H_n(x)e^{-x^2}$, i.e. use $\check{u}_n$

  Hermite polynomials

$\Rightarrow$ Functional differentiation is, up to basis change, usual differentiation

Note: How can $L^2[\mathbb{R}]$ have countable basis? Recall: $L^2[\mathbb{R}]$ consists not of functions, but of equivalence classes of functions.

Example application 1:

Schrödinger equation of QFT now well defined:

QM:	$\hat{q}_i$	$\hat{p}_i$	i	t
QFT:	$\hat{\phi}(x)$	$\hat{\pi}(x)$	x	t

QM: $\hat{H}(t) = \sum_{i=1}^{n} \frac{\hat{p}_i^{\,2}}{2} + V(\hat{q},t)$

all $\hat{q}_i$

Plays rôle of $V(\hat{q},t)$ although the first term is usually not considered to be part of the QFT's potential.

QFT: $\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{\pi}(x)^2}{2} + \underbrace{\frac{1}{2}\hat{\phi}(x)\left(m^2 - \Delta\right)\hat{\phi}(x) + W(\hat{\phi},t)}\, d^3x$

Example: $W(\hat{\phi}) = \frac{1}{4!}\hat{\phi}^4(x,t)$

In general: $W(\hat{\phi})$ also contains other fields

QM: Example of complete set of commuting s.adj. operators: $\{\hat{q}_i\}_{i=1}^{n}$

QFT: Example of complete set of commuting s.adj. operators: $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$

QM: The joint eigenbasis $\{|\{q_j\}_{j=1}^{m}\rangle\}$ of the $\{\hat{q}_j\}_{j=1}^{m}$ obeys:

$$\hat{q}_i\,|\{q_j\}_{j=1}^{m}\rangle = q_i\,|\{q_j\}_{j=1}^{m}\rangle$$

QFT: The joint eigenbasis $\{|\{\phi(x)\}_{x\in\mathbb{R}^3}\rangle\}$ of the $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$ obeys:

$$\hat{\phi}(y)\,|\{\phi(x)\}_{x\in\mathbb{R}^3}\rangle = \phi(y)\,|\{\phi(x)\}_{x\in\mathbb{R}^3}\rangle$$

QM: Wave function of a state $|\psi(t)\rangle \in \mathcal{H}$ in position eigenbasis:

$$\psi(\{q_j\}_{j=1}^{m}, t) = \langle\{q_j\}_{j=1}^{m}|\psi(t)\rangle \quad (\text{like } \psi(q) = \langle q|\psi\rangle)$$

Hilbert space of QFT, of course

QFT: Wave functional of a state $|\Psi(t)\rangle \in \mathcal{H}$ in field eigenbasis:

$$\Psi[\{\phi(x)\}_{x\in\mathbb{R}^3}, t] = \langle\{\phi(x)\}_{x\in\mathbb{R}^3}|\Psi(t)\rangle$$

Probability amplitude for finding function $\phi(x)$ when measuring $\hat{\phi}(x)$ at t.

Simplified notation:

QM: $\psi(q,t) = \langle q|\psi(t)\rangle$

QFT: $\Psi[\phi,t] = \langle\phi|\Psi(t)\rangle$

QM: Representation of $\hat{q}_i, \hat{p}_j$ obeying $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ in q eigenbasis:

$$\hat{q}_j: \quad \psi(q,t) \to q_j\,\psi(q,t)$$

$$\hat{p}_j: \quad \psi(q,t) \to -i\frac{\partial}{\partial q_j}\psi(q,t)$$

QFT: Representation of $\hat{\phi}(x), \hat{\pi}(y)$ obeying $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^3(x-y)$ in ϕ eigenbasis:

$$\hat{\phi}(x): \quad \Psi[\phi,t] \to \phi(x)\,\Psi[\phi,t]$$

$$\hat{\pi}(x): \quad \Psi[\phi,t] \to -i\frac{\delta}{\delta\phi(x)}\Psi[\phi,t]$$

Exercise: Verify that $\hat{\phi}(x), \hat{\pi}(x)$ obey the CCRs.

QM: Schrödinger equation:

$$i\frac{d}{dt}\psi(q,t) = \sum_{j=1}^{n} -\frac{1}{2}\frac{\partial^2}{\partial q_j^2}\psi(q,t) + V(q,t)\,\psi(q,t)$$

Recall: It is to be solved for all q

QFT: Schrödinger equation:

$$\boxed{i\frac{d}{dt}\Psi[\phi,t] = \int_{\mathbb{R}^3}\left(-\frac{1}{2}\frac{\delta^2}{\delta\phi(x)} + \frac{1}{2}\phi(x)(m^2-\Delta)\phi(x) + W(\phi(x),t)\,d^3x\right)\Psi[\phi,t]}$$

Recall: It is to be solved for all ϕ

Remark: With W it can be solved only perturbatively.

Exercise: Set $W=0$. Fourier transform to k variables in box regularization. Verify that the wave functional Ψ_0 of the vacuum state obtained before does obey the Schr. eqn.

Example application 2: The functional Legendre transform!

- **Motivation?** We will need to determine in curved space:

  What becomes of: $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$?

- **Problem?** Time is preferred coordinate in Hamiltonian formalism.

  * But the formalism must be coordinate system independent to fit general relativity (GR).

  * Now, for example, $\hat{\pi}(x,t) = \frac{d}{dt}\hat{\phi}(x,t)$ is not the same as $\hat{\pi}(x,\tau) = \frac{d}{d\tau}\hat{\phi}(x,\tau)$ for arbitrary $\tau(t)$:

$$\hat{\pi}(x,\tau) = \frac{d}{dt}\hat{\phi}(x,\tau(t)) = \frac{d}{d\tau}\hat{\phi}(x,\tau(t))\left(\frac{d\tau}{dt}\right) \neq \frac{d}{d\tau}\hat{\phi}(x,\tau) \; !$$

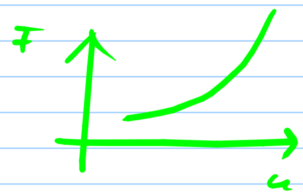
Strategy:

1. Transform to coordinate-independent Lagrange formalism.
2. Move from special to general relativity.
3. Transform GR result back to Hamilton formalism.
4. Apply 2nd quantization.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{SR, 1}^{st}\text{Q Hamiltonian formalism} \arrow[r, "\text{``Legendre transform''}", "\text{equivalence}"'] & \text{SR, 1}^{st}\text{Q. Lagrangian formalism} \arrow[d, "\text{allow curvature}"] \\
\text{GR, 1}^{st}\text{Q Hamiltonian formalism} \arrow[d, "\text{as outlined already}"] & \text{GR, 1}^{st}\text{Q Lagrangian formalism} \arrow[l, "\text{Legendre transform}"', "\text{equivalence}"] \arrow[d, dashed] \\
\text{GR, 2}^{nd}\text{Q Hamiltonian formalism} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{equivalence}"'] & \text{GR, 2}^{nd}\text{Q Lagrangian formalism (Path integral of QFT)}
\end{tikzcd}$$

The Legendre transform (LT):

- Assume given a function, $F(u)$.



- Define a new variable $w(u)$:

$$w(u) := \frac{dF}{du} \qquad \text{(I)}$$

- Assume that (I) can be solved to obtain:

$$u(w)$$

(that's ok if F is convex, say $F''(u) > 0$ for all u)

- The Legendre transform of F is a new function, G, of w:

$$F(u) \xrightarrow{LT} G(w)$$

- Namely: $\boxed{G(w) := w\, u(w) - F(u(w))}$

Proposition:

$$(LT)^2 = id$$

Proof:

- Define a new variable: $v(w) := \frac{\partial G(w)}{\partial w}$
- In fact:

$$v(w) = \frac{\partial}{\partial w}\left(w\,u(w) - \mathcal{F}(u(w))\right)$$

$$= u(w) + \cancel{w\,\frac{\partial u(w)}{\partial w}} - \cancel{\underbrace{\frac{\partial \mathcal{F}(u(w))}{\partial u}}_{=w}\frac{\partial u(w)}{\partial w}}$$

$$= u\ !$$

- Therefore LT^2 yields $\mathcal{F}(u) \xrightarrow{LT} G(w) \xrightarrow{LT} H(v)$ with:

$$H = vw - G = \underbrace{v}_{=u \text{ from just above}} w - (wu - \mathcal{F}) = \mathcal{F}$$

Example:

* Consider $f(a,b,c) := a e^{bc}$
* Find LT with respect to b (i.e. while treating a, c as "spectator variables":

$$f(a,b,c) \xrightarrow[b\to\beta]{LT} g(a,\beta,c)$$

* Define $\beta(a,b,c) := \frac{\partial f}{\partial b} = ace^{bc}$
* Invert: $b(a,\beta,c) = \frac{1}{c}\ln\frac{\beta}{ac}$
* Legendre transform: $f(a,b,c) \xrightarrow{LT} g(a,\beta,c)$

$$g(a,\beta,c) := \beta\, b(a,\beta,c) - f(a, b(a,\beta,c), c)$$

$$g(a,\beta,c) = \frac{\beta}{c}\ln\frac{\beta}{ac} - a e^{\frac{c}{c}\ln\frac{\beta}{ac}} = \frac{\beta}{c}\ln\frac{\beta}{ac} - \frac{\beta}{c}$$

Case of countably many variables:

- How to define
$$F(\{u_i\}) \xrightarrow{LT} G(\{w_i\}) \;?$$

- Define: $w_j := \frac{\partial F}{\partial u_j}$

- Assume we can invert to obtain:
$$u_j(\{w_i\})$$

- Define:
$$G(\{w_i\}) := \sum_j w_j u_j(\{w_i\}) - F(\{u_j(\{w_i\})\})$$

(we may also allow for spectator variables)

Case of uncountably many variables:

- How to define
$$F[\{u(x)\}_{x\in\mathbb{R}^n}] \xrightarrow{LT} G[\{w(x)\}_{x\in\mathbb{R}^n}] \;?$$

- Define: $w(x) := \frac{\delta F}{\delta u(x)}$

- Assume we can solve to obtain:
$$u(x, \{w(x')\}_{x'\in\mathbb{R}^n})$$

- Define:
$$G[\{w(x)\}_{x\in\mathbb{R}}] := \int_{\mathbb{R}^n} w(x)\, u(x, \{w(x')\}_{x'\in\mathbb{R}^n})\, d^n x - F[\{u(x, \{w(x')\})\}]$$

- Note: We still have that $LT \circ LT = id$.

Application to CM:

(CM: classical mechanics)

* Assume the Hamiltonian $H(q,p)$ is given.

* Hamilton equations for arbitrary $f(q,p)$:

Recall: Poisson bracket $\{q,p\}=1$

$$\dot f(q,p) = \{f(q,p), H(q,p)\}$$

See my notes to AMATH673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations <u>unchanged</u> while allowing q,p noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be $\hat q\hat p - \hat p\hat q = i\hbar$ and $\{\hat f,\hat g\} = \frac{1}{i\hbar}\{\hat f,\hat g\}$

* From this, one can prove the eqns of motion for q,p:

$$\dot q = \frac{\partial H(q,p)}{\partial p}, \qquad \dot p = -\frac{\partial H(q,p)}{\partial q} \qquad \text{(EoM)}$$

* Legendre transform:

The "Lagrangian"

$$H(q,p) \xrightarrow{LT} L(q,b) \qquad (q \text{ is spectator})$$

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* Example: $H(q,p) := \frac{p^2}{2} + V(q)$.

Then: $b \overset{L.T.}{:=} \frac{\partial H(q,p)}{\partial p} \overset{EoM}{=} \dot q$

$$\Rightarrow L(q,b) = b\,p(q,b) - H(q,p(q,b)) = \dot q\,p(q,\dot q) - H(q,p(q,\dot q)) = L(q,\dot q)$$

Proposition:

The equations of motion (EoM) now take the form:

$$b = \dot q \quad \text{and} \quad \frac{\partial L}{\partial q} = \frac{d}{dt}\frac{dL}{db} \qquad \text{(Euler Lagrange equation)}$$

Proof: Exercise

Example:

$$H = \frac{p^2}{2m} + \frac{\omega^2}{2}q^2 \xleftrightarrow{LT} L[q,b] = \frac{1}{2}\dot q^2 - \frac{\omega^2}{2}q^2$$

$$\dot q = \frac{p}{m}, \quad \dot p = -\omega^2 q \qquad\qquad -\omega^2 q = \ddot q, \quad b = \dot q$$

↙ classical (not conformal) field theory

Application to CFT:

- Assume Hamiltonian $H(\phi,\pi)$ is given.
- Hamilton equation for arbitrary $f(\phi,\pi)$:

$$\dot{f}(\phi,\pi,x,t) = \{f(\phi,\pi,x,t), H(\phi,\pi)\}$$

with: $$\{\phi(x,t),\pi(x',t)\} = \delta^3(x-x')$$

- This yields the eqns of motion:

$$\dot{\phi}(x,t) = \frac{\delta H}{\delta \pi(x,t)} \qquad \dot{\pi}(x,t) = -\frac{\delta H}{\delta \phi(x,t)} \qquad \text{(EoM)}$$

- Legendre Transform:

$$H(\phi,\pi) \xrightarrow{LT} L(\underset{\text{spectator}}{\phi}, \varsigma)$$

- Example: $H := \int \frac{1}{2}\pi(x,t)^2 + V(\phi(x))\, d^3x$

$$\varsigma(x,t) := \frac{\delta H}{\delta \pi(x,t)} \overset{\text{EoM}}{=} \dot{\phi}(x,t)$$

← Notice: this is because of the particular π^2 term in H. On curved space it will be different.

Thus:

$$L(\phi,\varsigma) = L(\phi,\dot{\phi})$$

$$= \int_{\mathbb{R}^3} \dot{\phi}(x,t)\,\pi(\phi,\dot{\phi},x,t)\, d^3x - H(\phi,\pi(\phi,\dot{\phi},x,t))$$

Proposition: The eqns of motion (EoM) are equivalent to:

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta \dot{\phi}(x,t)}$$

Exercise: Check

Euler Lagrange eqn.

Example:

$$H(\phi,\pi) = \int_{\mathbb{R}^3} \frac{\pi^2(x,t)}{2} + \frac{1}{2}\phi(x,t)\left(m^2-\Delta\right)\phi(x,t)\, d^3x$$

yields: $\quad \dot\phi(x,t) = \pi(x,t) \qquad \dot\pi(x,t) = (-m^2+\Delta)\,\phi(x,t)$

i.e: $\quad \ddot\phi - \Delta\phi + m^2\phi = 0 \qquad$ K.G. eqn.

After Legendre transform:

$$L(\phi,\dot\phi) = \int_{\mathbb{R}^3} \frac{\dot\phi^2(x,t)}{2} - \frac{1}{2}\phi(x,t)\left(m^2-\Delta\right)\phi(x,t)\, d^3x$$

yields directly: $\quad -(m^2-\Delta)\,\phi = \ddot\phi$

Remark: (see arxiv.0810.4293)

a) Solving a quantum theory is to do a Fourier transform.

b) The lowest order approximation is the Legendre transform.

c) The Legendre transform yields the solution to the classical theory.

a) Consider the path integral in QFT
(covered in detail later in this course)

$$e^{-iW[J]} = \int e^{i\overset{\uparrow}{\underset{\text{Classical action}}{S[\phi]}}}\; \underbrace{e^{-i\int J(x)\,\phi(x)\, d^4x}}_{\substack{\text{Fourier factors,}\\ \text{(one for each } x)}}\; \underbrace{D[\phi]}_{\prod_{x\in\mathbb{R}^4} d\phi(x)}$$

$J(x)$: "Source field"

(i.e. we integrate "over all fields ϕ")
(the proper function space is not known)

(To know $W[J]$ is to have solved the quantum field theory, because it yields all n-point correlation functions $G^{(n)}(x_1,\dots,x_n)$:
$$G^{(n)}(x_1,\dots,x_n) = \frac{\delta W[J]}{\delta J(x_1)\dots\delta J(x_n)}$$
)

$\Rightarrow$ $e^{iW[J]}$ is the Fourier transform of $e^{iS[\phi]}$.

b) The integrand contributes most where it is stationary:

$$e^{-iW[J]} \approx e^{iS[\phi]-i\int J\phi\, d^4x} \Bigg|_{\text{for that } \phi \text{ for which } \underbrace{\frac{\delta}{\delta\phi}\left(iS[\phi]-i\int J\phi\, d^4x\right)=0}_{\text{Condition of stationarity of the phase}}}$$

i.e.

$$W^{(approx)}[J] = \int J\phi\, d^4x - S[\phi] \Bigg|_{\text{where } \phi \text{ obeys } \frac{\delta S}{\delta\phi}(x) = J(x)}$$

i.e. $$W^{(approx)}[J] = \int J\,\phi[J]\, d^4x - S[\phi[J]] \Bigg|_{\text{where } \phi[J] \text{ follows from: } \frac{\delta S}{\delta\phi}(x) = J(x)}$$

i.e. it's the Legendre transform!

c) So what is knowing $W^{approx}[J]$ good for?

Consider $S^{total}[\phi] := S[\phi] - \int J\phi\, d^4x$.

As a classical action, it describes a classical field $\phi(x)$ driven by an external "driving force" $J(x)$:

$$\frac{\delta S^{total}}{\delta\phi} = 0 \text{ , i.e., } \boxed{\frac{\delta S}{\delta\phi}(x) = J(x) \qquad (EoM)}$$

To solve the classical equations of motion (EoM) is to find the field $\phi(x)$ for any given driving $J(x)$. This is what $W^{approx}[J]$ provides:

$$\boxed{\phi(x) = \frac{\delta W^{approx}[J]}{\delta J(x)}}$$

Because:

$(\text{Legendre transform})^2 = 1$